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Conic bundles over real formal power series field

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Abstract. We examine some properties of conic bundle rational surface over real formal power series field. We focus on the following problem: When does a conic bundle with prescribed degeneration data exist? We study also an algebraic counterpart of this problem (algebras, defined over a purely transcendental function field in one variable over real formal power series field).

1. Introduction

The main aim of the paper is to examine existence of conic bundle rational surfaces with prescribed degeneration data over real formal power series field. Let us first define the main object under consideration (see [1], [4], [5], [7], [8], [9], [12] for details).

Definition 1. [5] *A a conic bundle rational surface over a field K is a smooth, projective, geometrically integral K-variety X admitting a dominant K-morphism $\varphi\colon X \to \mathbb{P}^1_K$ whose generic fiber X_η isomorphic to a smooth conic.*

Such a fibration degenerates at a finite number of closed points $y_i \in \mathbb{P}^1$, and each degenerate fiber consists of a pair of smooth rational curves transversally intersecting at one point [8], [9]. Assume that φ is *relatively minimal*, i.e. no degenerate fiber can be blown down. Then each component of such a fiber is defined over a quadratic extension L_i of the residue field $K(y_i)$.

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Definition 2. *Let $\varphi\colon X \to \mathbb{P}^1_K$ be a relatively minimal conic bundle surface. The set of* **local invariants** *is defined as the collection of quadratic extensions $\{L_i/K(y_i)\}$, where $y_i \in \mathbb{P}^1$ are closed points at which φ degenerates, $K(y_i)$ is the residue field of y_i, and L_i is the field of definition of the components of the degenerate fiber at y_i.*

We would like to describe X in terms of this local information. The first question is existence:

Question A. Given a finite set $\{K(y_i), L_i\}$, where $y_i \in \mathbb{P}^1$ is a closed point and L_i is a quadratic extension of $K(y_i)$, does there exist a (relatively minimal) conic bundle $\varphi\colon X \to \mathbb{P}^1_K$ with local invariants $\{K(y_i), L_i\}$?

A natural obstruction to a positive answer to Question A becomes evident as soon as this question is translated into the language of quaternion algebras. Indeed, let Q denote a quaternion algebra over $F = K(x)$ corresponding to the generic fiber of X. Then Q ramifies precisely at the y_i's, and its ramification at y_i can be identified with a nonzero element of $K(y_i)^*/(K(y_i)^*)^2$ which, in turn, corresponds to some quadratic extension $L_i/K(y_i)$. More precisely, we have

Definition 3. *Let $F = K(x)$. Let $\mathcal{A}$ be a central simple F-algebra, and let $y \in \mathbb{P}^1_K$ be a closed point. The ramification of $\mathcal{A}$ at y is the element $\partial_y(\mathcal{A}) \in H^1(\gg_y, \mathbb{Q}/\mathbb{Z})$, where $\gg_y = Gal(\overline{K}/K(y))$ and $\partial_y\colon Br(F) \to H^1(\gg_y, \mathbb{Q}/\mathbb{Z})$ is the ramification map [15, Ch.II, App., §3]. We call the pair $(y, \partial_y(\mathcal{A}))$ with nonzero $\partial_y(\mathcal{A})$ a local invariant of $\mathcal{A}$.*

We need the following

Proposition 1. *([8], [9], [4])*

(i) There is a one-to-one correspondence between classes of birational (fiber-preserving) isomorphism of relatively minimal conic bundles $\varphi\colon X \to \mathbb{P}^1_K$ and isomorphism classes of quaternion algebras over $F = K(x)$;

(ii) Let $y \in \mathbb{P}^1_K$ be a closed point. There is a one-to-one correspondence between the following data:

- *collection of quadratic and trivial extensions of $K(y)$;*
- *$H^1(\gg_y, \mathbb{Z}/2)$;*
- *$K(y)^*/(K(y)^*)^2$.*

(iii) There is a one-to-one correspondence between closed points of $\mathbb{P}^1_K$ and discrete valuations of $K(x)$ trivial on K.

We thus get a one-to-one correspondence between the set of local invariants of a conic bundle $\varphi\colon X \to \mathbb{P}^1_K$ and the set of local invariants of the corresponding quaternion algebra.

Local invariants cannot take arbitrary values, but must satisfy the Faddeev reciprocity law.

Proposition 2. *([6], [15, Ch. II, App., §5]) There is an exact sequence*

$$0 \to {}_2Br(K) \to {}_2Br(F) \xrightarrow{\oplus\partial_y} \bigoplus_{y\in\mathbb{P}^1_K} H^1(K(y),\mathbb{Z}/2) \xrightarrow{cor} H^1(K,\mathbb{Z}/2) \to 0$$

(*here y runs over closed points of $\mathbb{P}^1_K$, $H^1(L,\cdot)$ is a shortening for $H^1(Gal(\overline{L}/L),\cdot)$, and cor is the sum of corestriction homomorphisms*).

Thus the sum of corestrictions of the values of local invariants must be 0, in order for us to have a positive answer to **Question A**.

Question A can be rephrased in terms of quaternion algebras:

Question B. Given a finite collection $\{y_i, d_i\}$, where $y_i \in \mathbb{P}^1$ is a closed point, $d_i \in K(y_i)^*/(K(y_i)^*)^2$, satisfying the Faddeev reciprocity law, does there exist a quaternion algebra Q over $F = K(x)$ which ramifies only at the y_i's, such that the residue of Q at y_i equals d_i for each i?

We call an element of $\bigoplus_{y\in\mathbb{P}^1_K} H^1(K(y),\mathbb{Z}/2)$ a system of invariants. For a finite collection $\{y_i, d_i\}$, where $y_i \in \mathbb{P}^1$ is a closed point and $d_i \in K(y_i)^*/(K(y_i)^*)^2$, there exists a corresponding element

$$\phi \in \bigoplus_{y\in\mathbb{P}^1_K} H^1(K(y),\mathbb{Z}/2).$$

The collection $\{y_i, d_i\}$ satisfies the Faddeev reciprocity law if $\phi \in \ker(\mathrm{cor})$.

Note that Faddeev's reciprocity law shows that if the finite local invariants of two algebras are the same, then their local invariants at infinity are also the same. More generally, if $\mathcal{A}_1$ and $\mathcal{A}_2$ agree at all but one closed rational point of ramification, then they agree at that point.

Furthermore, if $\mathcal{A}$ has local invariants of order 2, then $\mathcal{A}^{\otimes 2}$ has no nontrivial local invariants, and thus $\mathcal{A}^{\otimes 2}$ is isomorphic to some constant algebra $\mathcal{B}\otimes_K F$. Any central simple algebra of exponent 2 gives rise to its system of local invariants. Conversely by [6], there exists a central simple algebra $\mathcal{A}$ over F with local invariants $\{y_i, d_i\}$, and $\mathcal{A}$ is determined uniquely up to tensor multiplication by a constant algebra (i.e. algebra coming from a K-algebra). We can always choose $\mathcal{A}$ to have exponent 2 using classical results of Auslander–Brumer, Fein–Schacher and Rosset–Tate. In this way, we see that any system of local invariants corresponds to algebras of exponent 2.

Definition 4. *We call a collection $\{y_i, d_i\}$* **good** *if it is realizable as the system of local invariants of a quaternion algebra, and* **bad** *otherwise.*

Our fundamental question of which systems are good leads us to the following concept.

Definition 5. *Let K be a field, $F = K(x)$. We call central simple F-algebras $\mathcal{A}$ and $\mathcal{B}$ Faddeev equivalent if there is a K-algebra $\mathcal{C}$ such that $\mathcal{A}$ is Brauer equivalent to $\mathcal{B} \otimes_K \mathcal{C}$. We define the Faddeev index of $\mathcal{A}$ as the minimum of indices of algebras Faddeev equivalent to $\mathcal{A}$.*

Thus the Faddeev index of an algebra $\mathcal{A}$ of exponent 2 equals 2 if and only if the collection of local invariants of $\mathcal{A}$ is good.

Problem C. Compute the Faddeev index of any given $\mathcal{A}$.

Although Problem C seems to be out of reach for general K, for some fields the situation looks more optimistic. Namely, if K is a nondyadic $\mathfrak{p}$-adic field (= a finite extension of $\mathbb{Q}_p$, $p \neq 2$), the main result of Saltman's paper [14] gives a bound for the number of symbols in the Merkurjev–Suslin decomposition. If $[K : \mathbb{Q}_p] < \infty$, by [14, Th. 3.4], any $K(x)$-algebra $\mathcal{A}$ of prime exponent different from p is similar to the product of at most **two** cyclic algebras. Moreover, the appendix to [14] contains an example (due to Jacob and Tignol) showing that this estimate is sharp: The Jacob–Tignol algebra has exponent 2 and index 4.

Very little known when an algebra is Faddeev equivalent to a quaternion algebra in the case of a general field. The following result was appeared in ([10]) in a slightly different form.

Theorem 1. *Let $\mathcal{A}/K(x)$ be a central simple algebra of exponent 2, then $\mathcal{A}$ is Faddeev equivalent to a quaternion algebra, i.e. the corresponding system of local invariants is good, in the following cases of sets of ramification points*

- *two linear polynomials;*
- *a linear polynomial and the infinite point;*
- *an irreducible quadratic polynomial;*
- *three linear polynomials;*
- *two linear polynomials and the infinite point;*
- *a linear polynomial and an irreducible quadratic polynomial;*
- *an irreducible quadratic polynomial and the infinite point.*

Note that the latter result depends only on the set of ramification points but not on the types of ramification. If the degrees of ramification points or the cardinality of the set of ramification points is bigger then 3, no any other general results are known. The aim of the paper is to present some results on existence of conic bundles with prescribed degeneration data in terms of systems of good local invariants in cases of real formal power series field $\mathbb{R}((t))$.

In Section 2 we consider the algebraic counterpart of the above geometric problems. We prove the existence of a quaternion algebra with ramification at one or two polynomials which are sums of squares in $\mathbb{R}((t))[x]$.

In Section 3 we consider also the application of our results to the investigation of the Pfister conjecture about u-invariants of fields.

Below we fix the following notations and conventions. For an Abelian group A, the kernel of the multiplication by 2 is denoted by ${}_2A$. If R is a commutative ring, R^* denotes the group of units in R, and R^{*2} denotes the subgroup of squares in R^*. If $s \in R^*$, then for the brevity the class sR^{*2} will be denoted by the same symbol s.

We denote by $\mathrm{Br}(L)$ the Brauer group of a field L. For central simple L-algebras $\mathcal{A}, \mathcal{B}$ the equivalence $\mathcal{A} \sim \mathcal{B}$ will mean $[\mathcal{A}] = [\mathcal{B}]$ in $\mathrm{Br}(L)$ (here $[\mathcal{A}]$ is the element of $\mathrm{Br}(L)$ corresponding to $\mathcal{A}$); and we shall write $\mathcal{A} \sim 1$ if $[\mathcal{A}]$ is zero in $\mathrm{Br}(L)$. A quaternion L-algebra corresponding to a pair $a, b \in L^*$ is denoted by (a, b).

Let L be a field with a discrete valuation v. Then the ramification of a quaternion L-algebra (a, b) at v is defined by the square root of the residue of $(-1)^{v(a)v(b)} a^{v(b)}/b^{v(a)}$.

Let $K(x)$ be a pure transcendental extension of degree 1 over field K of zero characteristic. Recall the structure of discrete valuations of $F = K(x)$ trivial on K. Any such a valuation is of the following form. If it corresponds to a finite closed point y of $\mathbb{P}^1_K$, then there exists an irreducible monic polynomial $f(x) \in K[x]$ such that the valuation v_f with $f(x)$ as a uniformizer coincides with the valuation corresponding to y. The valuation corresponding to the infinite point is v_∞ with t^{-1} as a uniformizer.

The completions of $K(x)$ with respect to v_f and v_∞ are of the form $K(\theta)((f(x)))$ and $K((t^{-1}))$, respectively; here θ is a root of $f(x)$, and the embedding $K(x) \hookrightarrow K(\theta)((f(x)))$ sends t to some series $\tilde{t}$ such that

$$\tilde{t} \equiv \theta \pmod{f(x)}.$$

For an algebra $\mathcal{A}/F$ by $\mathcal{A}_f$, $\mathcal{A}_\infty$ we shall denote algebras $\mathcal{A} \otimes_K F_f$, $\mathcal{A} \otimes_K F_\infty$, where F_f (respectively F_∞) is the completion of F with respect to v_f (respectively v_∞).

Let $f(x), g(x) \in K[x]$, f irreducible and not dividing $g(x)$. Then the ramification of $(g(x), f(x))$ at f is $K(\theta)(\sqrt{g(\theta)})/K(\theta)$, where θ is a root of $f(x)$.

Let $\mathcal{A}$ be a central simple $\mathbb{R}((t))(x)$-algebra of exponent 2 and $f \in \mathbb{R}((t))[x]$ is a sum of squares. Note that the residue field of the valuation v_f is $\mathbb{C}((\sqrt[m]{t}))$ for some m. Since $\mathbb{C}((\sqrt[m]{t}))^*/(\mathbb{C}((\sqrt[m]{t}))^*)^2 = \{1, \sqrt[m]{t}\}$, then either $\mathcal{A}$ has no ramification at f or the ramification is uniquely defined.

2. Algebras over $\mathbb{R}((t))$ with special ramification

Let $\mathbb{R}((t))$ be the field of real formal power series. In this section we present an algebraic counterpart of the above geometric problem. We shall prove the existence of a quaternion algebra with ramification at one or two polynomials which are sums of squares in $\mathbb{R}((t))[x]$.

We shall need the following computational

Lemma 1. *Let f, g be relatively prime monic irreducible polynomials over $\mathbb{R}((t))$ which are sums of squares, and α, β respectively roots of f and g. Let also $\mathbb{R}((t))(\alpha) = \mathbb{C}((\sqrt[m]{t})), \mathbb{R}((t))(\beta) = \mathbb{C}((\sqrt[n]{t}))$, where $m, n \in \mathbb{N}$. Then a quaternion $\mathbb{R}((t))(x)$-algebra (f, g) is either trivial or ramified only at polynomials f, g.*

Proof. Let $L = \mathbb{R}((t))(\alpha, \beta)$, then $L = \mathbb{C}((\sqrt[k]{t}))$, where $k = \operatorname{lcm}(m, n)$. Indeed, let $m = dm_1, n = dn_1$, where $d = \gcd(m, n)$, then $k = dm_1n_1$. Since $m|k, n|k$, we have

$$\mathbb{R}((t))(\alpha, \beta) = \mathbb{C}((t))(\sqrt[m]{t}, \sqrt[n]{t}) \subset \mathbb{C}((t))(\sqrt[k]{t}) = \mathbb{C}((\sqrt[k]{t})).$$

There are $a, b \in \mathbb{Z}$ such that $an + bm = d$. Hence $a/m + b/n = d/(mn) = 1/k$. Thus $\sqrt[k]{t} = (\sqrt[m]{t})^a(\sqrt[n]{t})^b$, therefore

$$L = \mathbb{C}((t))(\sqrt[k]{t}) \subset \mathbb{C}((t))(\sqrt[m]{t}, \sqrt[n]{t}) = \mathbb{R}((t))(\alpha, \beta).$$

Hence $\mathbb{C}((\sqrt[k]{t})) = L$.

Let τ be the complex conjugation, σ an automorphism of L trivial on $\mathbb{C}$ such that $\sigma(\sqrt[k]{t}) = \varepsilon_k \sqrt[k]{t}$, where ε_k is a primitive k-th root of unity.

Note that $\sigma^m|_{\mathbb{R}((t))(\alpha)} = id$, $\sigma^n|_{\mathbb{R}((t))(\beta)} = id$. Indeed,

$$(\sqrt[n]{t})^{\sigma^n} = ((\sqrt[k]{t})^{m_1})^{\sigma^n} = ((\sqrt[k]{t})^{\sigma^n})^{m_1} = (\varepsilon_k^n \sqrt[k]{t})^{m_1} = \varepsilon_k^{nm_1}(\sqrt[k]{t})^{m_1} = \sqrt[n]{t}.$$

Since $(\sqrt[n]{t})^{\sigma^n} = \sqrt[n]{t}$, $\sigma^n|_{\mathbb{C}} = id$, then $\sigma^n|_{\mathbb{R}((t))(\beta)} = id$. In a similar way we obtain $\sigma^m|_{\mathbb{R}((t))(\alpha)} = id$. We shall show that

$$\{\sigma, \sigma^2, \ldots, \sigma^k, \tau\sigma, \ldots, , \tau\sigma^2, \tau\sigma^k\}$$

is the set of pairwise different automorphisms of L over $\mathbb{R}((t))$. If $1 \le i < j \le k$, then

$$(\sqrt[k]{t})^{\sigma^i} = \varepsilon_k^i \sqrt[k]{t} \neq \varepsilon_k^j \sqrt[k]{t} = (\sqrt[k]{t})^{\sigma^j},$$

$$(\sqrt[k]{t})^{\tau\sigma^i} = ((\sqrt[k]{t})^{\sigma^i})^{\tau} = (\varepsilon_k^i \sqrt[k]{t})^{\tau} = \varepsilon_k^{-i} \sqrt[k]{t} \neq$$

$$\varepsilon_k^{-j} \sqrt[k]{t} = (\varepsilon_k^j \sqrt[k]{t})^{\tau} = ((\sqrt[k]{t})^{\sigma^j})\tau = (\sqrt[k]{t})^{\tau\sigma^j}.$$

Besides, if $1 \le i \le k, 1 \le j \le k$, we have

$$(\sqrt{-1})^{\tau\sigma^i} = ((\sqrt{-1})^{\sigma^i})^{\tau} = (\sqrt{-1})^{\tau} = -\sqrt{-1} \neq \sqrt{-1} = (\sqrt{-1})^{\sigma^j}.$$

Hence $f(x) = \prod_{i=1}^m ((x - \alpha^{\sigma^i})(x - \alpha^{\tau\sigma^i}))$, $g(x) = \prod_{j=1}^n ((x - \beta^{\sigma^j})(x - \beta^{\tau\sigma^j}))$.

We have also that $\tau\sigma^i = \sigma^{k-i}\tau$ for all $i \in \mathbb{Z}$. Indeed,

$$\tau\sigma^i(\sqrt{-1}) = -\sqrt{-1} = \sigma^{k-i}\tau(\sqrt{-1}),$$

$$\tau\sigma^i(\sqrt[k]{t}) = \tau(\varepsilon_k^i \sqrt[k]{t}) = \varepsilon_k^{-i} \sqrt[k]{t} = \varepsilon_k^{k-i} \sqrt[k]{t} = \sigma^{k-i}(\sqrt[k]{t}) = \sigma^{k-i}\tau(\sqrt[k]{t}).$$

Note that for any $l, r \in \mathbb{Z}$

$$v_L(\beta - \alpha^{\sigma^{dl+r}}) = v_L(\beta - \alpha^{\sigma^r}), v_L(\beta - \alpha^{\tau\sigma^{dl+r}}) = v_L(\beta - \alpha^{\tau\sigma^r}),$$

where v_L is the valuation of L extending the valuation of $\mathbb{R}((t))$. Indeed,

$$v_L(\beta - \alpha^{\sigma^{dl+r}}) = v_L((\beta - \alpha^{\sigma^{dl+r}})^{\sigma^{-nla}}) = v_L(\beta^{(\sigma^n)^{-la}} - \alpha^{\sigma^{r+l(d-na)}})$$

$$= v_L(\beta - \alpha^{\sigma^{r+bml}}) = v_L(\beta - (\alpha^{(\sigma^m)^{bl}})^{\sigma^r}) = v_L(\beta - \alpha^{\sigma^r}),$$

$$v_L(\beta - \alpha^{\tau\sigma^{dl+r}}) = v_L((\beta - \alpha^{\tau\sigma^{dl+r}})^{\sigma^{-nla}})$$

$$= v_L(\beta^{(\sigma^n)^{-la}} - \alpha^{\tau\sigma^{r+l(d-na)}}) = v_L(\beta - \alpha^{\tau\sigma^{r+bml}}) = v_L(\beta - \alpha^{\sigma^{k-r-mbl}\tau})$$

$$= v_L(\beta - (\alpha^{(\sigma^m)^{-bl}})^{\sigma^{k-r}\tau}) = v_L(\beta - \alpha^{\sigma^{k-r}\tau}) = v_L(\beta - \alpha^{\tau\sigma^r}).$$

Then

$$v_L(f(\beta)) = v_L(\prod_{i=1}^m ((\beta - \alpha^{\sigma^i})(\beta - \alpha^{\tau\sigma^i})))$$

$$= v_L(\prod_{j=0}^{m_1-1} \prod_{r=1}^{d} ((\beta - \alpha^{\sigma^{dj+r}})(\beta - \alpha^{\tau\sigma^{dj+r}})))$$

$$= v_L(\prod_{j=0}^{m_1-1} \prod_{r=1}^{d} ((\beta - \alpha^{\sigma^r})(\beta - \alpha^{\tau\sigma^r})))$$

$$= v_L((\prod_{r=1}^{d}((\beta-\alpha^{\sigma^{dj+r}})(\beta-\alpha^{\tau\sigma^{dj+r}})))^{m_1})$$

$$= m_1 v_L(\prod_{r=1}^{d}((\beta-\alpha^{\sigma^{r}})(\beta-\alpha^{\tau\sigma^{r}}))).$$

In a similar way we obtain that

$$v_L(g(\beta)) = n_1 v_L(\prod_{r=1}^{d}((\alpha-\beta^{\sigma^{r}})(\alpha-\beta^{\tau\sigma^{r}}))).$$

Since

$$v_L(\alpha-\beta^{\tau\sigma^r}) = v_L((\alpha-\beta^{\tau\sigma^r})^{\tau\sigma^r}) = v_L(\alpha^{\tau\sigma^r}-\beta^{\tau(\sigma^r\tau)\sigma^r})$$

$$= v_L(\alpha^{\tau\sigma^r}-\beta^{\tau(\tau\sigma^{k-r})\sigma^r}) = v_L(\alpha^{\tau\sigma^r}-\beta^{\tau^2\sigma^k})$$

$$= v_L(\alpha^{\tau\sigma^r}-\beta) = v_L(\beta-\alpha^{\tau\sigma^r})$$

and

$$v_L(\alpha-\beta^{\sigma^r}) = v_L((\alpha-\beta^{\sigma^r})^{\sigma^{k-r}}) = v_L(\alpha^{\sigma^{k-r}}-\beta)$$

$$= v_L(\beta-\alpha^{\sigma^{k-r}}) = v_L(\beta-\alpha^{\sigma^{dm_1n_1-r}})$$

$$= v_L(\beta-\alpha^{\sigma^{d(m_1n_1-1)+(d-r)}}) = v_L(\beta-\alpha^{\sigma^{d-r}}),$$

we have

$$v_L(\prod_{r=1}^{d}((\alpha-\beta^{\sigma^{r}})(\alpha-\beta^{\tau\sigma^{r}}))) = v_L(\prod_{r=1}^{d}((\beta-\alpha^{\sigma^{d-r}})(\beta-\alpha^{\tau\sigma^{r}}))) =$$

$$v_L(\prod_{r=1}^{d}((\beta-\alpha^{\sigma^{r}})(\beta-\alpha^{\tau\sigma^{r}}))).$$

Hence

$$\frac{v_L(g(\alpha))}{n_1} = \frac{v_L(f(\beta))}{m_1}.$$

Let Γ, Γ_m, Γ_n be groups of values respectively of valuations of fields L, $\mathbb{C}((\sqrt[m]{t}))$, $\mathbb{C}((\sqrt[n]{t}))$ extending the valuation of $\mathbb{R}((t))$. Then

$$\Gamma = \frac{1}{n_1}\Gamma_m = \frac{1}{m_1}\Gamma_n.$$

The algebra (f,g) has ramification at $f \iff g(\alpha) \notin (\mathbb{C}((\sqrt[m]{t}))^*)^2 \iff v_L(g(\alpha)) \notin 2\Gamma_m \iff v_L(g(\alpha)) \notin 2n_1\Gamma \iff v_L(f(\beta)) \notin 2m_1\Gamma \iff f(\beta) \notin (\mathbb{C}((\sqrt[n]{t}))^*)^2 \iff$ the algebra (f,g) has ramification at g.

Note that $\mathbb{R}((t))(x)_\infty$ is a Pythagorean field, i.e. any sum of squares from $\mathbb{R}((t))(x)_\infty$ is a square. Then $f \in (\mathbb{R}((t))(x)^*_\infty)^2$ since f is a sum of squares. Hence $(f, g)_\infty \sim 1$. Thus if the algebra (f, g) has no ramification at f and g, then it is trivial. □

Lemma 2. *Let f_0 be a monic irreducible over $\mathbb{R}((t))$ polynomial which is a sum of two squares in $\mathbb{R}((t))[x]$, $\deg f_0 > 0$. Then there exist $f_1, \dots, f_n \in \mathbb{R}((t))[x]$, $n \in \mathbb{N}$ such that*

1) the algebra $(f_0, f_1) \otimes (f_1, f_2) \otimes \dots \otimes (f_{n-1}, f_n)$ is ramified only at f_0;

2) $f_n = t$ or f_n is an irreducible polynomial of positive degree such that $\mathbb{R}((t))(\theta_n)$ is a real field, where θ_n is a root of f_n;

3) $f_0, \dots, f_{n-1}$ are irreducible over $\mathbb{R}((t))$ polynomials which are sums of two squares in $\mathbb{R}((t))[x]$;

4) $\deg f_0 > \deg f_1 > \dots > \deg f_n$;

5) the algebras $(f_0, f_1), (f_1, f_2), \dots, (f_{n-1}, f_n)$ are nontrivial;

6) if $0 \le i < j - 1 \le n - 1$, then the algebra (f_i, f_j) is trivial.

Proof. Let θ_0 be a root of f_0. Since f_0 is a sum of squares over $\mathbb{R}((t))$, then $\mathbb{R}((t))(\theta) = \mathbb{C}((\sqrt[m]{t}))$ for some $m \in \mathbb{N}$. Recall that the ramification at f_0 is defined by any nonsquare element from $\mathbb{C}((\sqrt[m]{t}))$.

There is a polynomial $f \in \mathbb{R}((t))[x]$ such that $f(\theta_0) \notin (\mathbb{C}((\sqrt[m]{t}))^*)^2$, $\deg f < \deg f_0$. Among all such polynomials f choose a polynomial of the minimal degree and denote it by f_1. Note that f_1 is irreducible. Indeed, if $f_1 = g_1 g_2$, where $g_1, g_2 \in \mathbb{R}((t))[x], \deg g_1 > 0, \deg g_2 > 0$, then $g_1(\theta) \notin (\mathbb{C}((\sqrt[m]{t}))^*)^2$ or $g_2(\theta) \notin (\mathbb{C}((\sqrt[m]{t}))^*)^2$ since $f_1(\theta) \notin (\mathbb{C}((\sqrt[m]{t}))^*)^2$. Hence we have a contradiction with a choice of polynomial f_1 of minimal degree since $\deg g_i < \deg f_1$. Thus f_1 is irreducible.

Moreover, $(f_0, f_1)_\infty \sim 1$ by an argument analogous to that in the proof of Lemma 1.

Firstly assume that $\deg f_1 = 0$. Since $\mathbb{R}((t))^*/(\mathbb{R}((t))^*)^2 = \{\pm 1, \pm t\}$ and $(f_0, -1) \sim 1, (f_0, a^2) \sim 1$ for any $a \in \mathbb{R}((t)^*$, then one can obtain that $f_1 = t$. Thus if $\deg f_1 = 0$, then the algebra (f_0, f_1) is ramified only at f_0.

Now assume that $\deg f_1 > 0$. Let θ_1 be a root of the polynomial f_1. Consider the case, where $\mathbb{R}((t))(\theta_1)$ is real. Since $\mathbb{R}((t))(\theta_1)$ is real and f_0 is a sum of squares, then $f_0(\theta_1) \in (\mathbb{R}((t))(\theta_1)^*)^2$. Hence the algebra (f_0, f_1) is not ramified at f_1. Moreover, the latter algebra is not ramified at the infinite point by an argument analogous to that above. Hence the algebra (f_0, f_1) is ramified only at f_0.

If $\mathbb{R}((t))(\theta_1)$ is nonreal, then f_1 is a sum of squares. Hence by Lemma 1 the algebra (f_0, f_1) is ramified at f_0 and f_1. Since $\mathbb{R}(\theta_1)$ is nonreal, then $\deg f_1 \geq 2$. In the same way one can find an irreducible polynomial $f_2 \in K[x]$ of minimal degree such that $f_2(\theta_1) \notin (\mathbb{R}((t))(\theta_1)^*)^2$. Note that $\deg f_2 < \deg f_1$.

If $\mathbb{R}((t))(\theta_2)$ is real, where θ_2 is a root of polynomial f_2, then the algebra $(f_0, f_1) \otimes (f_1, f_2)$ ramifies only at f_0. If $\mathbb{R}((t))(\theta_2)$ is nonreal, then we can find a polynomial $f_3 \in \mathbb{R}((t))[x]$ of the minimal degree such that $f_3(\theta_2) \notin (\mathbb{R}((t))(\theta_2)^*)^2$, $\deg f_3 < \deg f_2$ and so on. Since $\deg f_{i+1} < \deg f_i$, then finally we shall obtain that either $\deg f_n = 0$ or the field $\mathbb{R}((t))(\theta_n)$ is real, where θ_n is a root of f_n. Then the algebra

$$(f_0, f_1) \otimes (f_1, f_2) \otimes \cdots \otimes (f_{n-1}, f_n)$$

ramifies only at f_0.

Now we shall show that the polynomials $f_1, \ldots, f_n$ satisfy to condition 2)...6). Properties 2)...5) are obvious from the construction of the polynomials f_i. Let us check the latter property. Consider the algebra (f_i, f_j), where $i+1 < j < n$. Since f_{i+1} is the polynomial of the minimal degree such that $f_{i+1}(\theta_i) \notin (\mathbb{R}((t))(\theta_i)^*)^2$, then $f_j(\theta_i) \in (\mathbb{R}((t))(\theta_i)^*)^2$. Hence the algebra (f_i, f_j) is not ramified at f_i. Then by Lemma 1 the latter algebra also is not ramified at f_j. In view of $(f_i, f_j)_\infty \sim 1$ one has that $(f_i, f_j) \sim 1$. □

Now it is possible to prove the main results.

Theorem 2. *Let $f \in \mathbb{R}((t))[x]$ be a monic irreducible polynomial which is a sum of two squares,* $\deg f > 0$. *Then there exists a quaternion algebra with ramification at f only.*

Proof. By Lemma 2 there exist polynomials $h_1, \ldots, h_n$ such that the algebra

$$\mathcal{A} = (f, h_1) \otimes \cdots \otimes (h_{n-1}, h_n)$$

has ramification only at f. Set $h_0 = 1$.

Let

$$\mathcal{B} = \begin{cases} (f \prod_{i=0}^{(n-1)/2} h_{2i}, \prod_{i=1}^{(n+1)/2} h_{2i-1}) & \text{if } n \equiv 1 (mod\ 2); \\ (f \prod_{i=0}^{n/2} h_{2i}, \prod_{i=1}^{n/2} h_{2i-1}) & \text{if } n \equiv 0 (mod\ 2). \end{cases}$$

Then $\mathcal{A}$ is Faddeev equivalent to $\mathcal{B}$. Indeed, let us consider the case where n is odd. We shall compute the ramification of $\mathcal{B}$. The algebra $\mathcal{B}$ is not ramified outside $\{f, h_1, \ldots, h_n\}$ and the infinite point.

Since $(f, h_i) \sim 1$ for $i > 1$, then $\mathcal{B}$ is ramified at f. Now consider the ramification at h_{2i}, $1 \leq i \leq (n-1)/2$. Note that by Lemma 2 $(h_{2i}, h_j) \sim 1$ if $j \neq 2i \pm 1$. Moreover, the algebra $(h_{2i}, h_{2i-1}h_{2i+1})$ has no ramification at h_{2i}. Hence $\mathcal{B}$ is not ramified at h_{2i}. Analogously, $\mathcal{B}$ is not ramified at h_{2i-1}, $1 \le i \le (n-1)/2$. Finally, since by Lemma 2 (f, h_n) and (h_{2i}, h_n), $1 \le i \le (n-1)/2$, have no ramification at h_n, then $\mathcal{B}$ is not ramified at h_n. Thus $\mathcal{B}$ and $\mathcal{A}$ have the same ramification at finite points. Hence by the reciprocity law, $\mathcal{B}$ has no ramification at the infinite point. This shows that $\mathcal{B}$ has ramification only at f.

The case where n is even can be treated in a similar way. □

Theorem 3. *Let $f_1, f_2 \in K[x]$ be monic irreducible polynomials which are sums of two squares,* $\deg f_1, \deg f_2 > 0$. *Then there exists a quaternion algebra with ramification only at f_1 and f_2.*

Proof. By Lemma 2 there exist polynomials $g_1, \dots, g_n$, $h_1, \dots, h_n$ such that the algebras

$$\mathcal{A}_1 = (f_1, h_1) \otimes \cdots \otimes (g_{n-1}, g_n),$$

$$\mathcal{A}_2 = (f, h_1) \otimes \cdots \otimes (h_{n-1}, h_n)$$

have ramification respectively only at f_1 and f_2. Set $g_0 = h_0 = 1$.

Firstly, consider the case where $(f_1, f_2) \not\sim 1$. By Lemma 1 the latter quaternion algebra is ramified at f_1 and f_2.

Let $(f_1, f_2) \sim 1$. Assume that there exists $k < n$ such that $(f_2, g_k) \not\sim 1$. Among all such k choose the minimal one. Then the algebra

$$(f_1, g_1) \otimes (g_1, g_2) \otimes \cdots \otimes (g_{k-1}, g_k)$$

is ramified only at f_2 and g_k. Moreover, for any $j < k$ the algebra (f_2, g_j) is trivial. By an analogous argument as in the proof of Theorem 2 one can prove that the algebra

$$\mathcal{B} = \begin{cases} (f_1 f_2 \prod_{i=0}^{(k-1)/2} g_{2i}, \prod_{i=1}^{(k+1)/2} g_{2i-1}) & \text{if } k \equiv 1 (mod\ 2); \\ (f_1 \prod_{i=0}^{k/2} g_{2i}, f_2 \prod_{i=1}^{k/2} g_{2i-1}) & \text{if } k \equiv 0 (mod\ 2) \end{cases}$$

has ramification only at f_1 and f_2.

In the same way one can obtain that if there is $l < m$ such that $(f_1, h_l) \not\sim 1$, then there exists a quaternion algebra with ramification only at f_1 and f_2.

Further we shall consider the case where $(f_2, g_k) \sim 1$, $(f_1, h_l) \sim 1$ for any $k < n, l < m$.

Assume that there are $k < n, l < m$ such that $(g_k, h_l) \not\sim 1$. Among all such pairs choose such that $(g_i, h_l) \sim 1$ for any $i < k$ and $(g_k, h_j) \sim 1$ for any $j < l$.

The algebra

$$(f_1, g_1) \otimes (g_1, g_2) \otimes \cdots \otimes (g_{k-1}, g_k)$$

is ramified only at points f_1, g_k and the algebra

$$(f_2, h_1) \otimes (h_1, h_2) \otimes \cdots \otimes (h_{l-1}, h_l)$$

is ramified only at points f_2, h_l. Moreover, $(f_1, h_j) \sim 1$ for any $j < l$, $(f_2, g_i) \sim 1$ for any $i < k$. Then one can prove that the algebra

$$\mathcal{B} = \begin{cases} (f_1 \prod_{i=0}^{(k-1)/2} g_{2i} \prod_{j=1}^{(l+1)/2} h_{2j-1}, f_2 \prod_{i=1}^{(k+1)/2} g_{2i-1} \prod_{j=0}^{(l-1)/2} h_{2j}) \\ \qquad\qquad \text{if } k \equiv 1 (mod\ 2), l \equiv 1 (mod\ 2); \\ (f_1 \prod_{i=0}^{k/2} g_{2i} F_2 \prod_{j=0}^{(l-1)/2} h_{2j}, \prod_{i=1}^{k/2} g_{2i-1} \prod_{i=1}^{(l+1)/2} h_{2j-1}) \\ \qquad\qquad \text{if } k \equiv 0 (mod\ 2), l \equiv 1 (mod\ 2); \\ (f_1 \prod_{i=0}^{(k-1)/2} g_{2i} F_2 \prod_{j=0}^{l/2} h_{2j}, \prod_{i=1}^{(k+1)/2} g_{2i-1} \prod_{i=1}^{l/2} h_{2j-1}) \\ \qquad\qquad \text{if } k \equiv 1 (mod\ 2), l \equiv 0 (mod\ 2); \\ (f_1 \prod_{i=0}^{k/2} g_{2i} \prod_{j=1}^{l/2} h_{2j-1}, f_2 \prod_{i=1}^{k/2} g_{2i-1} \prod_{j=0}^{l/2} h_{2j}) \\ \qquad\qquad \text{if } k \equiv 0 (mod\ 2), l \equiv 0 (mod\ 2). \end{cases}$$

is ramified only at f_1 and f_2.

We shall assume now that $(g_k, h_l) \sim 1$ for any $k < n, l < m$. Assume also that there is $k < n$ such that $(g_k, h_m) \not\sim 1$. Then the algebra (g_k, h_m) is ramified only at g_k. Among all such k choose such that $(g_i, h_m) \sim 1$ for any $i < k$. Hence the algebra

$$(f_1, g_1) \otimes \cdots \otimes (g_{k-1}, g_k) \otimes (g_k, h_m)$$

is is ramified only at f_1. Then the algebra

$$\mathcal{B} = \begin{cases} (f_1 \prod_{i=0}^{(k-1)/2} g_{2i} \prod_{j=1}^{(m+1)/2} h_{2j-1}, f_2 \prod_{i=1}^{(k+1)/2} g_{2i-1} \prod_{j=0}^{(m-1)/2} h_{2j}) \\ \qquad\qquad \text{if } k \equiv 1 (mod\ 2), m \equiv 1 (mod\ 2); \\ (f_1 \prod_{i=0}^{k/2} g_{2i} f_2 \prod_{j=0}^{(m-1)/2} h_{2j}, \prod_{i=1}^{k/2} g_{2i-1} \prod_{i=1}^{(m+1)/2} h_{2j-1}) \\ \qquad\qquad \text{if } k \equiv 0 (mod\ 2), m \equiv 1 (mod\ 2); \\ (f_1 \prod_{i=0}^{(k-1)/2} g_{2i} f_2 \prod_{j=0}^{m/2} h_{2j}, \prod_{i=1}^{(k+1)/2} g_{2i-1} \prod_{i=1}^{m/2} h_{2j-1}) \\ \qquad\qquad \text{if } k \equiv 1 (mod\ 2), m \equiv 0 (mod\ 2); \\ (f_1 \prod_{i=0}^{k/2} g_{2i} \prod_{j=1}^{m/2} h_{2j-1}, f_2 \prod_{i=1}^{k/2} g_{2i-1} \prod_{j=0}^{m/2} h_{2j}) \\ \qquad\qquad \text{if } k \equiv 0 (mod\ 2), m \equiv 0 (mod\ 2) \end{cases}$$

is ramified only at f_1 and f_2.

By the same way one can obtain that if there is $l < m$ such that $(g_n, h_l) \not\sim 1$, then there exists a quaternion algebra with ramification only at f_1 and f_2.

Finally, we shall assume that $(g_k, h_m) \sim 1$ and $(g_n, h_l) \sim 1$ for any $k < n, l < m$. Then the algebra

$$\mathcal{B} = \begin{cases} (f_1 \prod_{i=0}^{(n-1)/2} g_{2i} f_2 \prod_{j=0}^{(m-1)/2} h_{2j}, \prod_{i=1}^{(n+1)/2} g_{2i-1} \prod_{j=0}^{(m+1)/2} h_{2j-1}) \\ \qquad\qquad\qquad\qquad \text{if } n \equiv 1 (mod\ 2), m \equiv 1 (mod\ 2); \\ (f_1 \prod_{i=0}^{n/2} g_{2i} \prod_{j=1}^{(m+1)/2} h_{2j-1}, M_2 \prod_{i=1}^{n/2} H_{2i-1} \prod_{i=0}^{(m-1)/2} h_{2j-1}) \\ \qquad\qquad\qquad\qquad \text{if } n \equiv 0 (mod\ 2), m \equiv 1 (mod\ 2); \\ (f_1 \prod_{i=0}^{(n-1)/2} g_{2i} \prod_{j=1}^{m/2} h_{2j-1}, M_2 \prod_{i=1}^{(n+1)/2} g_{2i-1} \prod_{j=0}^{m/2} h_{2j}) \\ \qquad\qquad\qquad\qquad \text{if } n \equiv 1 (mod\ 2), m \equiv 0 (mod\ 2); \\ (f_1 \prod_{i=0}^{n/2} g_{2i} f_2 \prod_{j=0}^{m/2} h_{2j}, \prod_{i=1}^{n/2} g_{2i-1} \prod_{j=1}^{m/2} h_{2j-1}) \\ \qquad\qquad\qquad\qquad \text{if } n \equiv 0 (mod\ 2), m \equiv 0 (mod\ 2). \end{cases}$$

ramifies only at f_1 and f_2.

Thus in all cases there is a quaternion algebra with ramification only at f_1 and f_2. $\square$

At the end of the section we present a geometric reformulation of the theorems above.

Theorem 4. *Let $f \in \mathbb{R}((t))[x]$ be a monic irreducible polynomial which is a sum of two squares,* $\deg f > 0$. *Then there exists a conic bundle $\varphi\colon X \to \mathbb{P}^1_K$ with local invariant $\{\mathbb{R}((t))(x_f), L\}$, where $x_f \in \mathbb{P}^1$ is a closed point corresponding to f and L is a quadratic extension of $\mathbb{R}((t))(x_f)$.*

Theorem 5. *Let $f, g \in \{\mathbb{R}((t))(x_f)[x]$ be monic irreducible polynomials which are sums of two squares,* $\deg f_1, \deg f_2 > 0$. *Then there exists a conic bundle $\varphi\colon X \to \mathbb{P}^1_K$ with local invariants $\{\mathbb{R}((t))(x_f), L_1\}$ and $\{\mathbb{R}((t))(x_g), L_2\}$, where $x_f, x_g \in \mathbb{P}^1$ are closed points corresponding to f and g respectively and L_1, L_2 are quadratic extensions respectively of $\mathbb{R}((t))(x_f)$ and of $\mathbb{R}((t))(x_g)$.*

3. Ω-algebras over rational function fields

In this section we consider the application of results from the previous section to the investigation of Pfister conjecture about u-invariants of fields.

The u-invariant $u(F)$ of a field F is defined as $\sup\{\dim\varphi\}$, where φ is anisotropic quadratic form over F, $[\varphi]$ is a torsion element of the Witt group of F. Despite of the computations of $u(F)$ were done in many cases for special F so far there are a lot of interesting problems, which are still open. Among of them may be one of the most interesting is the following Pfister conjecture posed in [13].

Conjecture 1. Let F be an extension of transcendence degree m over some real closed field. Then $u(F) \leq 2^m$.

So far this conjecture does not prove even in the case $m = 2$. We shall be interested in case, where F is a real field. In that case Pfister conjecture can be reformulated as the statement of coincidence of the exponent and the index of the arbitrary Ω-algebra defined as follows.

Let Ω be the set of all orderings of the real field F. Then one has the following natural homomorphism

$$\psi : {}_2\mathrm{Br}(F) \to \prod_{\alpha \in \Omega} {}_2\mathrm{Br}(F_\alpha),$$

where F_α is the real closure of F with respect to an ordering α and ${}_2\mathrm{Br}(F)$, ${}_2\mathrm{Br}(F_\alpha)$ are 2-torsion parts of the Brauer groups of F and F_α respectively.

Definition 6. *An algebra $\mathcal{A}$ representing a nontrivial element of the kernel of ψ is called an Ω-algebra.*

In terms of the previous definition Conjecture 1 in case of real field F and $m = 2$ can be formulated as

Conjecture 2. Let F be an extension of transcendence degree 2 over some real closed field. Then the index of the arbitrary Ω-algebra is equal to 2.

Let us consider the case, where F is a pure transcendental extension of degree one of some curve C defined over field of real numbers $\mathbb{R}$, i.e. $F = \mathbb{R}(C)(x)$. Note that one can consider together with $\mathbb{R}(C)$ the family of completions $\mathbb{R}(C)_v$ of $\mathbb{R}(C)$ with respect to valuations v of $\mathbb{R}(C)$ trivial on $\mathbb{R}$. It is easy to see that any such completion is either of the form $\mathbb{C}((t))$ or $\mathbb{R}((t))$. From Conjecture 2 it follows

Conjecture 3. Let $F = \mathbb{R}((t))(x)$. Then the index of any Ω-algebra over F equals to 2.

As a corollary of Theorems 2 and 3 we can prove Conjecture 3 for some special class of Ω-algebras.

It was noted in [3] that the ramification locus of an Ω-algebra $\mathcal{A}/\mathbb{R}((t))(x)$ consists of polynomials which are sums of squares in $\mathbb{R}((t))(x)$ and in addition $\mathcal{A}_\infty \sim 1$. Moreover, in [3] it was shown that Conjecture 3 is valid for Ω-algebras with ramification loci consisting of either quadratic polynomials or polynomials with roots, which are not squares in their root fields over $\mathbb{R}((t))$. Another special cases for Ω-algebras were considered in [2].

Now we are in position to formulate the main result about Ω-algebras. Let f, g be monic irreducible polynomials over $\mathbb{R}((t))$ which are sums of squares in $\mathbb{R}((t))[x]$.

Theorem 6. *Let $\mathcal{A}$ be an Ω-algebra over $\mathbb{R}((t))(x)$. If the ramification locus of $\mathcal{A}/\mathbb{R}((t))(x)$ is either $\{f\}$ or $\{f, g\}$, then ind$(\mathcal{A}) = 2$.*

Proof. It is easy to see that the quaternion algebras $\mathcal{B}$ constructed in the proofs of Theorems 2 and 3 with ramification respectively at $\{f\}$ and $\{f, g\}$ are Ω-algebras. Hence $\mathcal{A} \sim \mathcal{B}$ and ind$(\mathcal{A}) = 2$. □

Remark 1. Summarizing we conclude that Conjecture 3 is valid for Ω-algebras such that its ramification loci are either $\{f\}$ or $\{f, g\}$.

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