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The commutator Hopf Galois extensions

George Szeto and Lianyong Xue

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Abstract. Let H be a finite dimentional Hopf algebra over a field k and H^* the dual Hopf algebra of H. Then a commutator right H^*-Galois extension B of B^H is characterized in terms of the smash product $B\#H$ and some relationships between such a B and the Hopf Galois Azumaya or Hopf Galois Hirata extensions are also given.

1. Introduction

Let H be a finite dimensional Hopf algebra over a field k, H^* the dual Hopf algebra of H, B a left module algebra over H with center C, and $V_B(B^H)$ the commutator subring of B^H in B. Let B be a right H^*-Galois extension of B^H, in [3], it was shown that $V_B(B^H)$ is a right H^*-Galois extension of Z where Z is the center of B^H if and only if $B \cong B^H \otimes_Z V_B(B^H)$ and $V_B(B^H)$ is a finitely generated and projective Z-module ([3], Lemma 2.8). Now we call B a commutator Hopf Galois extension of B^H if $V_B(B^H)$ is a right H^*-Galois extension of Z. When $H = (kG)^*$ for a finite automorphism group G of B, a commutator Hopf Galois extension is a commutator Galois extension as studied in [6]. The purpose of the present paper is to characterize a commutator Hopf Galois extension in terms of the Hirata separability of the smash product $B\#H$ over B^H. Moreover, a relationship is also given between the class of commutator Hopf Galois extensions and the class of Hopf Galois Azumaya extensions as studied in [4] and [7] or the class of Hopf Galois Hirata extensions, respectively, where B is called a right H^*-Galois

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Hirata extension of B^H if B is a right H^*-Galois extension and a Hirata separable extension of B^H as given in [8].

2. Basic definitions and notations

Throughout, H denotes a finite dimensional Hopf algebra over a field k with comultiplication Δ and counit ε, H^* the dual Hopf algebra of H, B a left H-module algebra, C the center of B, $B^H = \{b \in B \,|\, hb = \varepsilon(h)b$ for all $h \in H\}$ which is called the H-invariants of B, and $B\#H$ the smash product of B with H where $B\#H = B \otimes_k H$ such that for all $b\#h$ and $b'\#h'$ in $B\#H$, $(b\#h)(b'\#h') = \sum b(h_1 b')\#h_2 h'$ where $\Delta(h) = \sum h_1 \otimes h_2$. B is called a right H^*-Galois extension of B^H if B is a right H^*-comodule algebra with structure map $\rho : B \longrightarrow B \otimes_k H^*$ such that $\beta : B \otimes_{B^H} B \longrightarrow B \otimes_k H^*$ is a bijection where $\beta(a \otimes b) = (a \otimes 1)\rho(b)$.

For a subring A of B with the same identity 1, we denote the commutator subring of A in B by $V_B(A)$. we call B a separable extension of A if there exist $\{a_i, b_i$ in B, $i = 1, 2, ..., m$ for some integer $m\}$ such that $\sum a_i b_i = 1$, and $\sum b a_i \otimes b_i = \sum a_i \otimes b_i b$ for all b in B where $\otimes$ is over A. An Azumaya algebra is a separable extension of its center. A ring B is called a Hirata separable extension of A if $B \otimes_A B$ is isomorphic to a direct summand of a finite direct sum of B as a B-bimodule. A right H^*-Galois extension B is called a commutator right H^*-Galois extension if $V_B(B^H)$ is a right H^*-Galois extension of the center of B^H, a right H^*-Galois Azumaya extension if B is separable over B^H which is an Azumaya algebra over C^H, and a right H^*-Galois Hirata extension if B is also a Hirata separable extension of B^H. Throughout, an H^*-Galois extension means a right H^*-Galois extension unless it is stated otherwise.

3. The commutator Hopf Galois extensions

In this section, we shall characterize a commutator Hopf Galois extension in terms of the smash product $B\#H$.

Theorem 3.1. *Let B be an H^*-Galois extension of B^H and Z the center of B^H. Then the following statements are equivalent:*

(1) B is a commutator H^*-Galois extension of B^H,

(2) $B\#H$ is a projective Hirata separable extension of B^H and contains B^H as a direct summand as a B^H-bimodule,

(3) $B\#H \cong B^H \otimes_Z (V_B(B^H)\#H)$ and $V_B(B^H)\#H$ is a finitely generated and projective Z-module, and

(4) $V_B(B^H)\#H$ is an Azumaya Z-algebra.

Proof. (1) $\Longrightarrow$ (4) Since B is a commutator H^*-Galois extension of B^H, $V_B(B^H)$ is an H^*-Galois extension of Z. Hence $V_B(B^H)\#H \cong \mathrm{Hom}_Z(V_B(B^H), V_B(B^H))$ such that $V_B(B^H)$ is a finitely generated and projective Z-module ([3], Theorem 1.7). But Z is commutative with 1, so $V_B(B^H)$ is a progenerator of Z. Thus $\mathrm{Hom}_Z(V_B(B^H), V_B(B^H))$ is an Azumaya Z-algebra ([1], Proposition 4.1 on page 56); and so $V_B(B^H)\#H$ is an Azumaya Z-algebra.

(4) $\Longrightarrow$ (1) Since $V_B(B^H)\#H$ is an Azumaya Z-algebra, finitely generated and projective over $V_B(B^H)$, $V_B(B^H)\#H$ is a Hirata separable extension of $V_B(B^H)$ ([1], Theorem 1). But $V_B(B^H)$ is a progenerator of $V_B(B^H)$, so $V_B(B^H)$ is a progenerator of $V_B(B^H)\#H$ ([2], Lemma). Hence $V_B(B^H)$ is an H^*-Galois extension of $(V_B(B^H))^H$ ([4], Theorem 2.2). Noting that $(V_B(B^H))^H = V_{B^H}(B^H) = Z$, we have that B is a commutator H^*-Galois extension of B^H.

(1) $\Longrightarrow$ (2) By hypothesis, $V_B(B^H)$ is an H^*-Galois extension of Z, so $B \cong B^H \otimes_Z V_B(B^H)$ such that $V_B(B^H)$ is a finitely generated and projective Z-module ([3], Lemma 2.8). Noting that the above isomorphism is also a ring isomorphism, we have that $B\#H \cong (B^H \otimes_Z V_B(B^H))\#H \cong B^H \otimes_Z (V_B(B^H)\#H)$. But by (1) $\Longrightarrow$ (4), $V_B(B^H)\#H$ is an Azumaya Z-algebra, so $B^H \otimes_Z (V_B(B^H)\#H)$ is a projective Hirata separable extension of B^H. Also Z is a direct summand of $V_B(B^H)\#H$ as a Z-bimodule ([1], Lemma 3.1, page 51), so B^H is a direct summand of $B^H \otimes_Z (V_B(B^H)\#H)$ as a B^H-bimodule. Thus (2) holds.

(2) $\Longrightarrow$ (4) Since $B\#H$ is a projective Hirata separable extension of B^H and contains B^H as a direct summand as a B^H-bimodule, $V_{B\#H}(B^H)$ is a separable Z-algebra such that $V_{B\#H}(V_{B\#H}(B^H)) = B^H$ ([5], Proposition 1.3 and Proposition 1.4). This implies that the center of $B\#H$, B^H, and $V_{B\#H}(B^H)$ are the same Z. Thus $V_B(B^H)\#H$ $(= V_{B\#H}(B^H))$ is an Azumaya Z-algebra.

(1) $\Longrightarrow$ (3) Since $V_B(B^H)$ is an H^*-Galois extension of Z, $B \cong B^H \otimes_Z V_B(B^H)$ such that $V_B(B^H)$ is a finitely generated and projective Z-module ([3], Lemma 2.8). Noting that this isomorphism is also a ring isomorphism, we have that $B\#H \cong (B^H \otimes_Z V_B(B^H))\#H \cong B^H \otimes_Z (V_B(B^H)\#H)$ such that $V_B(B^H)\#H$ is a finitely generated and projective Z-module by (1) $\Longrightarrow$ (4).

(3) $\Longrightarrow$ (1) Since H is a finite dimensional Hopf algebra over a field k, the statement (3) implies that $B \cong B^H \otimes_Z V_B(B^H)$ and $V_B(B^H)$ is a finitely generated and projective Z-module. Thus $V_B(B^H)$ is an H^*-Galois extension of Z ([3], Lemma 2.8). □

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4. Three kinds of Hopf Galois extensions

In section 2, we call B an H^*-Galois Hirata extension of B^H if B is an H^*-Galois extension and a Hirata separable extension of B^H. In [8], it was shown that an H^*-Galois extension B of B^H is a Hirata separable extension of B^H if and only if $V_B(B^H)$ is a left H-Galois extension of C ([8], Theorem 2.6). In this section, we shall show that an H^*-Galois Hirata extension of B^H equal to B^{H^*} is a commutator H^*-Galois extension of B^H. Moreover, an equivalent condition is given under which a commutator H^*-Galois extension of B^H becomes an H^*-Galois Hirata extension of B^H or an H^*-Galois Azumaya extension of B^H.

Theorem 4.1.

(1) If B is an H^*-Galois Azumaya extension of B^H, then B is a commutator H^*-Galois extension of B^H.

(2) If B is an H^*-Galois Hirata extension of B^H equal to B^{H^*}, then B is a commutator H^*-Galois extension of B^H.

Proof. (1) Let B be an H^*-Galois Azumaya extension of B^H. Then, by Lemma 4.1 in [4], $V_B(B^H)$ is an H^*-Galois extension of $(V_B(B^H))^H$. Since $(V_B(B^H))^H = V_{B^H}(B^H) = Z$, B is a commutator H^*-Galois extension of B^H.

(2) Let B be an H^*-Galois Hirata extension of B^H. Then $V_B(B^H)$ is a left H-Galois extension of C ([8], Theorem 2.6). Hence $V_B(B^H)$ is a finitely generated and projective C-module. Since C is commutative with 1, $V_B(B^H)$ is a progenerator of C. But then the argument used in the proof of Lemma 2.8 in [3] can be applied here. Let $\Omega = \mathrm{Hom}_C(V_B(B^H), V_B(B^H))$. Then, as a left Ω-module, $B \cong V_B(B^H) \otimes_C \mathrm{Hom}_\Omega(V_B(B^H), B)$. Noting that $\mathrm{Hom}_\Omega(V_B(B^H), B) \cong B^{H^*}$ by $f \longrightarrow f(1)$ for each f in $\mathrm{Hom}_\Omega(V_B(B^H), B)$, we have that $B \cong V_B(B^H) \otimes_C B^{H^*}$. But $V_B(B^H)$ is a left H-Galois extension of C, so $V_B(V_B(B^H)) = B^H = B^{H^*}$ ([8], Lemma 2.5). Hence $B \cong V_B(B^H) \otimes_C B^H$. Moreover, since $V_B(V_B(B^H)) = B^H$ again, the centers of B^H and B are the same C. Therefore $V_B(B^H)$ is an H^*-Galois extension of C ([3], Lemma 2.8), that is, B is a commutator H^*-Galois extension of B^H. □

Next two theorems relate the class of commutator H^*-Galois extensions to the class of H^*-Galois Azumaya extensions and the class of H^*-Galois Hirata extensions.

Theorem 4.2. *Let B be a commutator H^*-Galois extension of B^H. Then B is separable over Z, the center of B^H, if and only if B is an H^*-Galois Azumaya extension of B^H.*

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Proof. ($\Longrightarrow$) Since $B \cong B^H \otimes_Z V_B(B^H)$ as a Z-algebra such that $V_B(B^H)$ is an H^*-Galois extension of Z which is a direct summand of $V_B(B^H)$ as a Z-module ([3], Proposition 1.9 and Corollary 1.10). By hypothesis, B is separable over Z, so B^H is separable over Z ([1], Corollary 1.9 on page 45). Thus B^H is an Azumaya Z-algebra. Moreover, since $B \cong B^H \otimes_Z V_B(B^H)$ again, $Z = C^H$. Therefore B is a separable and H^*-Galois extension of B^H which is an Azumaya C^H-algebra; that is, B is an H^*-Galois Azumaya extension of B^H ([4], Theorem 3.4).

($\Longleftarrow$) B is an H^*-Galois Azumaya extension of B^H, so B is a separable extension of B^H and B^H is an Azumaya C^H-algebra. Hence B is a separable extension of Z ($= C^H$) by the transitivity of separable extensions. □

Theorem 4.3. *Let B be a commutator H^*-Galois extension of B^H. Then $V_B(B^H)$ is an Azumaya Z-algebra if and only if B is an H^*-Galois Hirata extension of B^H.*

Proof. ($\Longrightarrow$) Since $V_B(B^H)$ is an H^*-Galois extension of Z, $B \cong B^H \otimes_Z V_B(B^H)$ as a Z-algebra. But $V_B(B^H)$ is an Azumaya Z-algebra, so B is a Hirata separable extension of B^H. Hence B is an H^*-Galois Hirata extension of B^H.

($\Longleftarrow$) Since B is an H^*-Galois extension of B^H, B is a finitely generated and projective B^H-module. By hypothesis, B is a projective Hirata separable extension of B^H and contains B^H as a direct summand as a B^H-bimodule by Theorem 3.1-(2). But then $V_B(B^H)$ is a separable C-algebra and $V_B(V_B(B^H)) = B^H$ ([5], Proposition 1.3 and Proposition 1.4). This implies that the center of B, B^H, and $V_B(B^H)$ are the same Z ($= C = C^H$). Thus $V_B(B^H)$ is an Azumaya Z-algebra. □

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CONTACT INFORMATION

G. Szeto Department of Mathematics Bradley University Peoria, Illinois 61625, USA
E-Mail: szeto@hilltop.bradley.edu

L. Xue Department of Mathematics Bradley University Peoria, Illinois 61625, USA
E-Mail: lxue@hilltop.bradley.edu

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