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Free rectangular tribands

Yuliia V. Zhuchok

Abstract. We introduce the notion of a rectangular triband, construct a free rectangular triband and describe its structure.

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1 Introduction

Recall that a vector space (set) T equipped with three binary associative operations $\dashv$, $\vdash$ and $\perp$ that satisfy the following axioms: $(x \dashv y) \dashv z = x \dashv (y \vdash z)\,(T1)$, $(x \vdash y) \dashv z = x \vdash (y \dashv z)\,(T2)$, $(x \dashv y) \vdash z = x \vdash (y \vdash z)\,(T3)$, $(x \dashv y) \dashv z = x \dashv (y \perp z)\,(T4)$, $(x \perp y) \dashv z = x \perp (y \dashv z)\,(T5)$, $(x \dashv y) \perp z = x \perp (y \vdash z)\,(T6)$, $(x \vdash y) \perp z = x \vdash (y \perp z)\,(T7)$, $(x \perp y) \vdash z = x \vdash (y \vdash z)\,(T8)$ for all $x, y, z \in T$, is called a trialgebra (trioid) [1]. So, the notion of a trialgebra is based on the notion of a trioid and all results obtained for trioids can be applied to trialgebras. This connection between trioids and trialgebras gives a motivation for studying trioids. Another reason for our interest in trioids is their connection with dimonoids [2, 3]. For a general introduction and basic theory see [1, 4].

The first step in the study of idempotent semigroups has been made by David McLean [5] who used rectangular bands for the description of the structure of an arbitrary band. Rectangular dimonoids (rectangular dibands) first appeared in the researches of the structure of dibands of subdimonoids (see [6]). Using rectangular dibands, a structure theorem on idempotent dimonoids was given in [7]. The free rectangular diband was constructed in [8].

In this paper we introduce the notion of a rectangular triband and give examples of rectangular tribands (Lemmas 1–4). We also construct a free rectangular triband (Theorem 1), describe its structure (Theorems 3–4) and the automorphism group (Lemma 5). As a consequence of Theorem 2, some least congruences on free rectangular tribands are described (Corollary 2).

2 Preliminaries

A nonempty subset A of a trioid $(T, \dashv, \vdash, \perp)$ is called a subtrioid if for any $a, b \in T$, a, $b \in A$ it follows that $a \dashv b$, $a \vdash b, a \perp b \in A$. An idempotent semigroup

S is called a rectangular band if

$$xyx = x \tag{1}$$

for all $x, y \in S$. It is clear that in any rectangular band the identity

$$xyz = xz \tag{2}$$

holds.

A trioid $(T, \dashv, \vdash, \perp)$ is called an idempotent trioid or a triband [9] if semigroups $(T, \dashv)$, $(T, \vdash)$ and $(T, \perp)$ are idempotent semigroups. A trioid $(T, \dashv, \vdash, \perp)$ will be called a rectangular trioid or a rectangular triband, if semigroups $(T, \dashv)$, $(T, \vdash)$ and $(T, \perp)$ are rectangular bands.

Note that the class of all rectangular tribands is a subvariety of the variety of all trioids. A trioid which is free in the variety of rectangular tribands will be called a free rectangular triband.

Recall the definition of a triband of subtrioids which was introduced in [9].

If $f : T_1 \to T_2$ is a homomorphism of trioids, then the corresponding congruence on T_1 will be denoted by Δ_f.

Let S be an arbitrary trioid, J be some idempotent trioid and let $\alpha : S \to J : x \mapsto x\alpha$ be a homomorphism. Then every class of the congruence Δ_α is a subtrioid of the trioid S, and the trioid S itself is a union of such trioids S_ξ, $\xi \in J$, that

$$x\alpha = \xi \Leftrightarrow x \in S_\xi = \Delta_\alpha^x = \{t \in S \,|\, (x, t) \in \Delta_\alpha\},$$

$$S_\xi \dashv S_\varepsilon \subseteq S_{\xi \dashv \varepsilon}, \quad S_\xi \vdash S_\varepsilon \subseteq S_{\xi \vdash \varepsilon}, \quad S_\xi \perp S_\varepsilon \subseteq S_{\xi \perp \varepsilon},$$

$$\xi \neq \varepsilon \Rightarrow S_\xi \cap S_\varepsilon = \varnothing.$$

In this case we say that S is decomposable into a triband of subtrioids (or S is a triband J of subtrioids S_ξ $(\xi \in J)$). If J is an idempotent semigroup (band), then we say that S is a band J of subtrioids S_ξ $(\xi \in J)$. If J is a commutative band, then we say that S is a semilattice J of subtrioids S_ξ $(\xi \in J)$. If J is a left (right) zero semigroup, then we say that S is a left (right) band J of subtrioids S_ξ $(\xi \in J)$.

Observe that the notion of a triband of subtrioids generalizes the notion of a diband of subdimonoids [6] (see also [7]) and the notion of a band of semigroups [10].

Recall that a nonempty set D equipped with two binary associative operations $\dashv$ and $\vdash$ satisfying the axioms $(T1)-(T3)$ is called a dimonoid [2, 3]. If $D = (D, \dashv, \vdash)$ is a dimonoid, then the trioid $(D, \dashv, \vdash, \dashv)$ (respectively, $(D, \dashv, \vdash, \vdash)$) will be denoted by $(D)^{\dashv}$ (respectively, $(D)^{\vdash}$). It is clear that $(D)^{\dashv}$ and $(D)^{\vdash}$ are different as trioids but they coincide as dimonoids.

Consider the following dimonoids from [8] which will be used in Section 4.

Let X be an arbitrary nonempty set. Let $X_{\ell z} = (X, \dashv)$, $X_{rz} = (X, \vdash)$, $X_{rb} = X_{\ell z} \times X_{rz}$ be a left zero semigroup, a right zero semigroup and a rectangular band, respectively. By [8] $X_{\ell z, rz} = (X, \dashv, \vdash)$ is the free left zero and right zero dimonoid (or the free left and right diband).

Define operations $\dashv$ and $\vdash$ on X^2 by

$$(x,y) \dashv (a,b) = (x,b), \quad (x,y) \vdash (a,b) = (a,b)$$

for all $(x,y),(a,b) \in X^2$. By [8] $(X^2, \dashv, \vdash)$ is a free (rb, rz)-dimonoid. It is denoted by $X_{rb,rz}$.

Define operations $\dashv$ and $\vdash$ on X^2 by

$$(x,y) \dashv (a,b) = (x,y), \quad (x,y) \vdash (a,b) = (x,b)$$

for all $(x,y),(a,b) \in X^2$. By [8] $(X^2, \dashv, \vdash)$ is a free $(\ell z, rb)$-dimonoid. It is denoted by $X_{\ell z,rb}$.

Define operations $\dashv$ and $\vdash$ on X^3 by

$$(x_1,x_2,x_3) \dashv (y_1,y_2,y_3) = (x_1,x_2,y_3),$$

$$(x_1,x_2,x_3) \vdash (y_1,y_2,y_3) = (x_1,y_2,y_3)$$

for all $(x_1,x_2,x_3),(y_1,y_2,y_3) \in X^3$. The algebra $(X^3, \dashv, \vdash)$ is denoted by $FRct(X)$. According to Theorem 1 from [8] $FRct(X)$ is a free rectangular diband.

As usual, $\mathbb{N}$ denotes the set of all positive integers.

3 Rectangular tribands

In this section we give new examples of rectangular tribands and construct a free rectangular triband of an arbitrary rank.

We first give examples of rectangular tribands.

It is immediate to prove the following three lemmas.

Let $I_n = \{1, 2, ..., n\}$, $n > 1$, and let $\{X_i\}_{i \in I_n}$ be a family of arbitrary nonempty sets X_i, $i \in I_n$. Define operations $\dashv$, $\vdash$ and $\perp$ on $\prod_{i \in I_3} X_i$ by

$$(a_1,b_1,c_1) \dashv (a_2,b_2,c_2) = (a_1,b_1,c_1),$$

$$(a_1,b_1,c_1) \vdash (a_2,b_2,c_2) = (a_1,b_2,c_2),$$

$$(a_1,b_1,c_1,) \perp (a_2,b_2,c_2) = (a_1,b_1,c_2)$$

for all $(a_1,b_1,c_1),(a_2,b_2,c_2) \in \prod_{i \in I_3} X_i$. It is clear that $(\prod_{i \in I_3} X_i, \perp, \vdash)$ is a rectangular diband [8] and $(\prod_{i \in I_3} X_i, \dashv)$ is a left zero semigroup.

Lemma 1. $(\prod_{i \in I_3} X_i, \dashv, \vdash, \perp)$ *is a rectangular triband.*

If $X_i = X$ for all $i \in I_3$, then denote the algebra $(\prod_{i \in I_3} X_i, \dashv, \vdash, \perp)$ by $X_{lz,rd}$.

Define operations $\dashv$, $\vdash$ and $\perp$ on $\prod_{i \in I_3} X_i$ by

$$(a_1,b_1,c_1) \dashv (a_2,b_2,c_2) = (a_1,b_1,c_2),$$

$$(a_1,b_1,c_1) \vdash (a_2,b_2,c_2) = (a_2,b_2,c_2),$$

$$(a_1,b_1,c_1,) \perp (a_2,b_2,c_2) = (a_1,b_2,c_2)$$

for all $(a_1,b_1,c_1),(a_2,b_2,c_2) \in \prod_{i \in I_3} X_i$. It is clear that $(\prod_{i \in I_3} X_i, \dashv, \perp)$ is a rectangular diband [8] and $(\prod_{i \in I_3} X_i, \vdash)$ is a right zero semigroup.

Lemma 2. $(\prod_{i\in I_3} X_i, \dashv, \vdash, \perp)$ *is a rectangular triband.*

If $X_i = X$ for all $i \in I_3$, then denote the algebra $(\prod_{i\in I_3} X_i, \dashv, \vdash, \perp)$ by $X_{rd,rz}$.

Define operations $\dashv$, $\vdash$ and $\perp$ on $\prod_{i\in I_2} X_i$ by

$$(a_1, b_1)\dashv(a_2, b_2) = (a_1, b_1), \quad (a_1, b_1)\vdash(a_2, b_2) = (a_2, b_2),$$

$$(a_1, b_1)\perp(a_2, b_2) = (a_1, b_2)$$

for all $(a_1, b_1), (a_2, b_2) \in \prod_{i\in I_2} X_i$. It is clear that $(\prod_{i\in I_2} X_i, \dashv, \vdash)$ is a left zero and right zero dimonoid [8] and $(\prod_{i\in I_2} X_i, \perp)$ is a rectangular band.

Lemma 3. $(\prod_{i\in I_2} X_i, \dashv, \vdash, \perp)$ *is a rectangular triband.*

If $X_i = X$ for all $i \in I_2$, then denote $(\prod_{i\in I_2} X_i, \dashv, \vdash, \perp)$ by $X^{rb}_{lz,rz}$. Note that the trioid $X^{rb}_{lz,rz}$ was first constructed in [9].

Define operations $\dashv$, $\vdash$ and $\perp$ on $\prod_{i\in I_{2k}} X_i$, where $k \in \mathbb{N}$, by

$$(x_1, x_2, ..., x_{2k})\dashv(y_1, y_2, ..., y_{2k}) = (x_1, x_2, ..., x_{2k-1}, y_{2k}),$$

$$(x_1, x_2, ..., x_{2k})\vdash(y_1, y_2, ..., y_{2k}) = (x_1, y_2, ..., y_{2k}),$$

$$(x_1, x_2, ..., x_{2k})\perp(y_1, y_2, ..., y_{2k}) = (x_1, x_2, ..., x_k, y_{k+1}, ..., y_{2k})$$

for all $(x_1, x_2, ..., x_{2k}), (y_1, y_2, ..., y_{2k}) \in \prod_{i\in I_{2k}} X_i$.

Lemma 4. *For any $k > 1$, $\left(\prod_{i\in I_{2k}} X_i, \dashv, \vdash, \perp\right)$ is a rectangular triband.*

Proof. By Lemma 4 from [8] $(\prod_{i\in I_{2k}} X_i, \dashv, \vdash, \perp)$ satisfies the axioms $(T1) - (T3)$ of a trioid and the associativity of operations $\dashv, \vdash$. For all $(x_1, x_2, ..., x_{2k})$, $(y_1, y_2, ..., y_{2k}), (z_1, z_2, ..., z_{2k}) \in \prod_{i\in I_{2k}} X_i$ obtain

$$((x_1, x_2, ..., x_{2k})\perp(y_1, y_2, ..., y_{2k}))\perp(z_1, z_2, ..., z_{2k}) =$$

$$= (x_1, x_2, ..., x_k, y_{k+1}, ..., y_{2k})\perp(z_1, z_2, ..., z_{2k}) =$$

$$= (x_1, x_2, ..., x_k, z_{k+1}, ..., z_{2k}) = (x_1, x_2, ..., x_{2k})\perp(y_1, y_2, ..., y_k, z_{k+1}, ..., z_{2k}) =$$

$$= (x_1, x_2, ..., x_{2k})\perp((y_1, y_2, ..., y_{2k})\perp(z_1, z_2, ..., z_{2k})),$$

$$((x_1, x_2, ..., x_{2k})\dashv(y_1, y_2, ..., y_{2k}))\dashv(z_1, z_2, ..., z_{2k}) =$$

$$= (x_1, x_2, ..., x_{2k-1}, y_{2k})\dashv(z_1, z_2, ..., z_{2k}) =$$

$$= (x_1, x_2, ..., x_{2k-1}, z_{2k}) = (x_1, x_2, ..., x_{2k})\dashv(y_1, y_2, ..., y_k, z_{k+1}, ..., z_{2k}) =$$

$$= (x_1, x_2, ..., x_{2k})\dashv((y_1, y_2, ..., y_{2k})\perp(z_1, z_2, ..., z_{2k})),$$

$$((x_1, x_2, ..., x_{2k})\perp(y_1, y_2, ..., y_{2k}))\dashv(z_1, z_2, ..., z_{2k}) =$$

$$= (x_1, x_2, ..., x_k, y_{k+1}, ..., y_{2k})\dashv(z_1, z_2, ..., z_{2k}) =$$

$$= (x_1, x_2, ..., x_k, y_{k+1}, ..., y_{2k-1}, z_{2k}) = (x_1, x_2, ..., x_{2k})\perp(y_1, y_2, ..., y_{2k-1}, z_{2k}) =$$

$$= (x_1, x_2, ..., x_{2k}) \perp ((y_1, y_2, ..., y_{2k}) \dashv (z_1, z_2, ..., z_{2k})),$$

$$((x_1, x_2, ..., x_{2k}) \dashv (y_1, y_2, ..., y_{2k})) \perp (z_1, z_2, ..., z_{2k}) =$$

$$= (x_1, x_2, ..., x_{2k-1}, y_{2k}) \perp (z_1, z_2, ..., z_{2k}) =$$

$$= (x_1, x_2, ..., x_k, z_{k+1}, ..., z_{2k}) = (x_1, x_2, ..., x_{2k}) \perp (y_1, z_2, ..., z_{2k}) =$$

$$= (x_1, x_2, ..., x_{2k}) \perp ((y_1, y_2, ..., y_{2k}) \vdash (z_1, z_2, ..., z_{2k})),$$

$$((x_1, x_2, ..., x_{2k}) \vdash (y_1, y_2, ..., y_{2k})) \perp (z_1, z_2, ..., z_{2k}) = (x_1, y_2, ..., y_{2k}) \perp (z_1, z_2, ..., z_{2k}) =$$

$$= (x_1, y_2, ..., y_k, z_{k+1}, ..., z_{2k}) = (x_1, x_2, ..., x_{2k}) \vdash (y_1, y_2, ..., y_k, z_{k+1}, ..., z_{2k}) =$$

$$= (x_1, x_2, ..., x_{2k}) \vdash ((y_1, y_2, ..., y_{2k}) \perp (z_1, z_2, ..., z_{2k})),$$

$$((x_1, x_2, ..., x_{2k}) \perp (y_1, y_2, ..., y_{2k})) \vdash (z_1, z_2, ..., z_{2k}) =$$

$$= (x_1, x_2, ..., x_k, y_{k+1}, ..., y_{2k}) \vdash (z_1, z_2, ..., z_{2k}) =$$

$$= (x_1, z_2, ..., z_{2k}) = (x_1, x_2, ..., x_{2k}) \vdash (y_1, z_2, ..., z_{2k}) =$$

$$= (x_1, x_2, ..., x_{2k}) \vdash ((y_1, y_2, ..., y_{2k}) \vdash (z_1, z_2, ..., z_{2k})).$$

Thus, $(\prod_{i \in I_{2k}} X_i, \dashv, \vdash, \perp)$ satisfies the axioms $(T4) - (T8)$ of a trioid and the associativity of $\perp$ and so, it is a trioid. Obviously, $(\prod_{i \in I_{2k}} X_i, \dashv, \vdash, \perp)$ is idempotent. Show that it is a rectangular triband. We have

$$(x_1, x_2, ..., x_{2k}) \dashv (y_1, y_2, ..., y_{2k}) \dashv (x_1, x_2, ..., x_{2k}) =$$

$$= (x_1, x_2, ..., x_{2k-1}, y_{2k}) \dashv (x_1, x_2, ..., x_{2k}) = (x_1, x_2, ..., x_{2k}),$$

$$(x_1, x_2, ..., x_{2k}) \vdash (y_1, y_2, ..., y_{2k}) \vdash (x_1, x_2, ..., x_{2k}) =$$

$$= (x_1, y_2, ..., y_{2k}) \vdash (x_1, x_2, ..., x_{2k}) = (x_1, x_2, ..., x_{2k}),$$

$$(x_1, x_2, ..., x_{2k}) \perp (y_1, y_2, ..., y_{2k}) \perp (x_1, x_2, ..., x_{2k}) =$$

$$= (x_1, x_2, ..., x_k, y_{k+1}, ..., y_{2k}) \perp (x_1, x_2, ..., x_{2k}) = (x_1, x_2, ..., x_{2k}).$$

Thus, $(\prod_{i \in I_{2k}} X_i, \dashv, \vdash, \perp)$ is a rectangular triband. □

Obviously, operations of $(\prod_{i \in I_2} X_i, \dashv, \vdash, \perp)$ coincide and it is a rectangular band.

Let X be an arbitrary nonempty set. We denote the trioid $(X^4, \dashv, \vdash, \perp)$ by $FRT(X)$.

The main result of this section is the following

Theorem 1. *$FRT(X)$ is a free rectangular triband.*

Proof. By Lemma 4 $FRT(X)$ is a rectangular triband. Let $(T, \dashv', \vdash', \perp')$ be an arbitrary rectangular trioid and $\sigma : X \to T$ be an arbitrary map. Define the map

$$\tau : \ FRT(X) \to (T, \dashv', \vdash', \perp') :$$

$$(a, b, c, d) \mapsto (a, b, c, d)\tau = (a\sigma \vdash' b\sigma) \perp' (c\sigma \dashv' d\sigma).$$

In order to prove that τ is a homomorphism we will use axioms of a trioid and the identities (1), (2). One can get

$$((a_1, b_1, c_1, d_1) \dashv (a_2, b_2, c_2, d_2))\tau = (a_1, b_1, c_1, d_2)\tau = (a_1\sigma \vdash' b_1\sigma) \perp' (c_1\sigma \dashv' d_2\sigma) =$$
$$= (a_1\sigma \vdash' b_1\sigma) \perp' ((c_1\sigma \dashv' d_1\sigma) \dashv' (c_2\sigma \dashv' d_2\sigma)) =$$
$$= ((a_1\sigma \vdash' b_1\sigma) \perp' (c_1\sigma \dashv' d_1\sigma)) \dashv' (c_2\sigma \dashv' d_2\sigma) =$$
$$= ((a_1\sigma \vdash' b_1\sigma) \perp' (c_1\sigma \dashv' d_1\sigma)) \dashv' (a_2\sigma \vdash' b_2\sigma) \dashv' (c_2\sigma \dashv' d_2\sigma) =$$
$$= ((a_1\sigma \vdash' b_1\sigma) \perp' (c_1\sigma \dashv' d_1\sigma)) \dashv'$$
$$\dashv' ((a_2\sigma \vdash' b_2\sigma) \perp' (c_2\sigma \dashv' d_2\sigma)) = (a_1, b_1, c_1, d_1)\tau \dashv' (a_2, b_2, c_2, d_2)\tau,$$
$$((a_1, b_1, c_1, d_1) \vdash (a_2, b_2, c_2, d_2))\tau = (a_1, b_2, c_2, d_2)\tau = (a_1\sigma \vdash' b_2\sigma) \perp' (c_2\sigma \dashv' d_2\sigma) =$$
$$= a_1\sigma \vdash' (b_2\sigma \perp' (c_2\sigma \dashv' d_2\sigma)) = a_1\sigma \vdash' ((b_2\sigma \vdash' a_2\sigma \vdash' b_2\sigma) \perp' (c_2\sigma \dashv' d_2\sigma)) =$$
$$= a_1\sigma \vdash' ((b_2\sigma \vdash' (a_2\sigma \vdash' b_2\sigma)) \perp' (c_2\sigma \dashv' d_2\sigma)) =$$
$$= a_1\sigma \vdash' (b_2\sigma \vdash' ((a_2\sigma \vdash' b_2\sigma) \perp' (c_2\sigma \dashv' d_2\sigma))) =$$
$$= a_1\sigma \vdash' ((a_2\sigma \vdash' b_2\sigma) \perp' (c_2\sigma \dashv' d_2\sigma)) =$$
$$= a_1\sigma \vdash' b_1\sigma \vdash' (c_1\sigma \dashv' d_1\sigma) \vdash' ((a_2\sigma \vdash' b_2\sigma) \perp' (c_2\sigma \dashv' d_2\sigma)) =$$
$$= (a_1\sigma \vdash' b_1\sigma) \vdash' ((c_1\sigma \dashv' d_1\sigma) \vdash' ((a_2\sigma \vdash' b_2\sigma) \perp' (c_2\sigma \dashv' d_2\sigma))) =$$
$$= ((a_1\sigma \vdash' b_1\sigma) \perp' (c_1\sigma \dashv' d_1\sigma)) \vdash' ((a_2\sigma \vdash' b_2\sigma) \perp' (c_2\sigma \dashv' d_2\sigma)) =$$
$$= (a_1, b_1, c_1, d_1)\tau \vdash' (a_2, b_2, c_2, d_2)\tau,$$
$$((a_1, b_1, c_1, d_1) \perp (a_2, b_2, c_2, d_2))\tau = (a_1, b_1, c_2, d_2)\tau = (a_1\sigma \vdash' b_1\sigma) \perp' (c_2\sigma \dashv' d_2\sigma) =$$
$$= ((a_1\sigma \vdash' b_1\sigma) \perp' (c_1\sigma \dashv' d_1\sigma)) \perp' ((a_2\sigma \vdash' b_2\sigma) \perp' (c_2\sigma \dashv' d_2\sigma)) =$$
$$= (a_1, b_1, c_1, d_1)\tau \perp' (a_2, b_2, c_2, d_2)\tau.$$

Thus, τ is a homomorphism and $FRT(X)$ is free. $\square$

Corollary 1. *The free rectangular triband $FRT(X)$ generated by a finite set X is finite. Specifically, if $|X| = n$, then $|FRT(X)| = n^4$.*

Denote the symmetric group on X by $\Im[X]$ and the automorphism group of a trioid M by $Aut\, M$. It is not difficult to see that the set $\{(a, a, a, a) \mid a \in X\}$ is generating for $FRT(X)$. From here obtain the following description of the automorphism group of the free rectangular triband.

Lemma 5. $Aut\, FRT(X) \cong \Im[X]$.

4 Decompositions of $FRT(X)$

In this section we describe the structure of free rectangular tribands and characterize some least congruences on free rectangular tribands.

For all $i, j \in X$ put

$$\Lambda_{(i)} = \{(a,b,c,d) \in FRT(X) \mid a = i\} \ ,$$

$$\Lambda_{[i]} = \{(a,b,c,d) \in FRT(X) \mid d = i\} \ ,$$

$$\Lambda_{(i,j)} = \{(a,b,c,d) \in FRT(X) \mid (a,d) = (i,j)\} \ .$$

The following theorem gives decompositions of $FRT(X)$ into bands of subtrioids.

Theorem 2. *Let $FRT(X)$ be a free rectangular triband. Then*

(i) $FRT(X)$ is a left band X_{lz} of subtrioids $\Lambda_{(i)}, i \in X_{lz}$, such that $\Lambda_{(i)} \cong X_{rd,rz}$ for every $i \in X_{lz}$;

(ii) $FRT(X)$ is a right band X_{rz} of subtrioids $\Lambda_{[i]}, i \in X_{rz}$, such that $\Lambda_{[i]} \cong X_{lz,rd}$ for every $i \in X_{rz}$;

(iii) $FRT(X)$ is a rectangular band X_{rb} of subtrioids $\Lambda_{(i,j)}, (i,j) \in X_{rb}$, such that $\Lambda_{(i,j)} \cong X^{rb}_{lz,rz}$ for every $(i,j) \in X_{rb}$.

Proof. (i) By Theorem 1 the map $\pi_{lz} : FRT(X) \to X_{lz} : (a,b,c,d) \mapsto a$ is a homomorphism. Then $\Lambda_{(i)}, i \in X_{lz}$, is a class of $\Delta_{\pi_{lz}}$ which is a subtrioid of $FRT(X)$. It is immediate to check that for every $i \in X_{lz}$ the map

$$\Lambda_{(i)} \to X_{rd,rz} : (i,b,c,d) \mapsto (b,c,d)$$

is an isomorphism.

(ii) By Theorem 1 the map $\pi_{rz} : FRT(X) \to X_{rz} : (a,b,c,d) \to d$ is a homomorphism. Then $\Lambda_{[i]}$, $i \in X_{rz}$, is a class of $\Delta_{\pi_{rz}}$ which is a subtrioid of $FRT(X)$. It is clear that for every $i \in X_{rz}$ the map

$$\Lambda_{[i]} \to X_{lz,rd} : (a,b,c,i) \mapsto (a,b,c)$$

is an isomorphism.

(iii) By Theorem 1 the map $\pi_{rb} : FRT(X) \to X_{rb} : (a,b,c,d) \to (a,d)$ is a homomorphism. Then $\Lambda_{(i,j)}, (i,j) \in X_{rb}$, is a class of $\Delta_{\pi_{rb}}$ which is a subtrioid of $FRT(X)$. It can be shown that for every $(i,j) \in X_{rb}$ the map

$$\Lambda_{(i,j)} \to X^{rb}_{lz,rz} : (i,b,c,j) \mapsto (b,c)$$

is an isomorphism. □

If ρ is a congruence on a trioid $(T, \dashv, \vdash, \perp)$ such that operations of $(T, \dashv, \vdash, \perp)/\rho$ coincide and it is a left zero semigroup (respectively, right zero semigroup, rectangular band, semilattice), then we say that ρ is a left zero congruence (respectively, right zero congruence, rectangular band congruence, semilattice congruence).

From Theorem 2 we obtain

Corollary 2. *Let $FRT(X)$ be a free rectangular triband. Then*
(i) $\Delta_{\pi_{lz}}$ is the least left zero congruence on $FRT(X)$;
(ii) $\Delta_{\pi_{rz}}$ is the least right zero congruence on $FRT(X)$;
(iii) $\Delta_{\pi_{rb}}$ is the least rectangular band congruence on $FRT(X)$.

Proof. (i) It is well-known that every left zero semigroup is a free left zero semigroup. By Theorem 2 (i) we obtain (i).

The proofs of (ii) and (iii) are similar. □

From Theorem 5 [11] it follows that any rectangular triband is semilattice indecomposable, i.e. the least semilattice congruence on a rectangular triband coincides with the universal relation on this trioid.

For all $i, j, k \in X$ put

$$\Lambda_{(i,j,k)} = \{(a, b, c, d) \in FRT(X) \mid (a, b, c) = (i, j, k)\} ,$$

$$\Lambda_{[i,j,k]} = \{(a, b, c, d) \in FRT(X) \mid (b, c, d) = (i, j, k)\} ,$$

$$\Lambda_{[i,j]} = \{(a, b, c, d) \in FRT(X) \mid (b, c) = (i, j)\} .$$

The following theorem gives decompositions of $FRT(X)$ into tribands of subsemigroups.

Theorem 3. *Let $FRT(X)$ be a free rectangular triband. Then*
(i) $FRT(X)$ is a triband $X_{lz,rd}$ of subsemigroups $\Lambda_{(i,j,k)}, (i, j, k) \in X_{lz,rd}$, such that $\Lambda_{(i,j,k)} \cong X_{rz}$ for every $(i, j, k) \in X_{lz,rd}$;
(ii) $FRT(X)$ is a triband $X_{rd,rz}$ of subsemigroups $\Lambda_{[i,j,k]}, (i, j, k) \in X_{rd,rz}$, such that $\Lambda_{[i,j,k]} \cong X_{lz}$ for every $(i, j, k) \in X_{rd,rz}$;
(iii) $FRT(X)$ is a triband $X^{rb}_{lz,rz}$ of subsemigroups $\Lambda_{[i,j]}, (i, j) \in X^{rb}_{lz,rz}$, such that $\Lambda_{[i,j]} \cong X_{rb}$ for every $(i, j) \in X^{rb}_{lz,rz}$.

Proof. (i) By Theorem 1 the map

$$\pi_{lz,rd} : FRT(X) \to X_{lz,rd} : (a, b, c, d) \mapsto (a, b, c)$$

is a homomorphism. Then $\Lambda_{(i,j,k)}, (i, j, k) \in X_{lz,rd}$, is a class of $\Delta_{\pi_{lz,rd}}$ which is a subtrioid of $FRT(X)$. If $(a_1, b_1, c_1, d_1), (a_2, b_2, c_2, d_2) \in \Lambda_{(i,j,k)}$, then $(a_1, b_1, c_1) = (a_2, b_2, c_2) = (i, j, k)$ and

$$(a_1, b_1, c_1, d_1) \dashv (a_2, b_2, c_2, d_2) = (a_1, b_1, c_1, d_2) = (i, j, k, d_2),$$

$$(a_1, b_1, c_1, d_1) \vdash (a_2, b_2, c_2, d_2) = (a_1, b_2, c_2, d_2) = (i, j, k, d_2),$$

$$(a_1, b_1, c_1, d_1) \perp (a_2, b_2, c_2, d_2) = (a_1, b_1, c_2, d_2) = (i, j, k, d_2).$$

Hence operations of $\Lambda_{(i,j,k)}$ coincide and so, it is a semigroup. It is easy to cheek that for every $(i, j, k) \in X_{lz,rd}$ the map $\Lambda_{(i,j,k)} \to X_{rz} : (i, j, k, d) \mapsto d$ is an isomorphism.

(ii) By Theorem 1 the map

$$\pi_{rd,rz} : FRT(X) \to X_{rd,rz} : (a, b, c, d) \mapsto (b, c, d)$$

is a homomorphism. Then $\Lambda_{[i,j,k]}, (i,j,k) \in X_{rd,rz}$, is a class of $\Delta_{\pi_{rd,rz}}$which is a subtrioid of $FRT(X)$. If $(a_1,b_1,c_1,d_1),(a_2,b_2,c_2,d_2) \in \Lambda_{[i,j,k]}$, then $(b_1,c_1,d_1) = (b_2,c_2,d_2) = (i,j,k)$ and

$$(a_1,b_1,c_1,d_1)\dashv(a_2,b_2,c_2,d_2) = (a_1,b_1,c_1,d_2) = (a_1,i,j,k),$$

$$(a_1,b_1,c_1,d_1)\vdash(a_2,b_2,c_2,d_2) = (a_1,b_2,c_2,d_2) = (a_1,i,j,k),$$

$$(a_1,b_1,c_1,d_1)\perp(a_2,b_2,c_2,d_2) = (a_1,b_1,c_2,d_2) = (a_1,i,j,k).$$

Hence operations of $\Lambda_{[i,j,k]}$ coincide and so, it is a semigroup. One can check that for every $(i,j,k) \in X_{rd,rz}$ the map $\Lambda_{[i,j,k]} \to X_{lz} : (a,i,j,k) \mapsto a$ is an isomorphism.

(iii) By Theorem 1 the map

$$\pi^{rb}_{lz,rz} : FRT(X) \to X^{rb}_{lz,rz} : (a,b,c,d) \mapsto (b,c)$$

is a homomorphism. Then $\Lambda_{[i,j]}, (i,j) \in X^{rb}_{lz,rz}$, is a class of $\Delta_{\pi^{rb}_{lz,rz}}$ which is a subtrioid of $FRT(X)$. If $(a_1,b_1,c_1,d_1),(a_2,b_2,c_2,d_2) \in \Lambda_{[i,j]}$, then $(b_1,c_1) = (b_2,c_2) = (i,j)$ and

$$(a_1,b_1,c_1,d_1)\dashv(a_2,b_2,c_2,d_2) = (a_1,b_1,c_1,d_2) = (a_1,i,j,d_2),$$

$$(a_1,b_1,c_1,d_1)\vdash(a_2,b_2,c_2,d_2) = (a_1,b_2,c_2,d_2) = (a_1,i,j,d_2),$$

$$(a_1,b_1,c_1,d_1)\perp(a_2,b_2,c_2,d_2) = (a_1,b_1,c_2,d_2) = (a_1,i,j,d_2).$$

Hence operations of $\Lambda_{[i,j]}$ coincide and so, it is a semigroup. An immediate verification shows that for every $(i,j) \in X^{rb}_{lz,rz}$ the map $\Lambda_{[i,j]} \to X_{rb} : (a,i,j,d) \mapsto (a,d)$ is an isomorphism. □

For all $i,j,k \in X$ put

$$V_{(i)} = \{(a,b,c,d) \in FRT(X) \mid b = i\}\ ,$$

$$V_{[i]} = \{(a,b,c,d) \in FRT(X) \mid c = i\}\ ,$$

$$V_{(i,j,k)} = \{(a,b,c,d) \in FRT(X) \mid (a,b,d) = (i,j,k)\}\ ,$$

$$V_{[i,j,k]} = \{(a,b,c,d) \in FRT(X) \mid (a,c,d) = (i,j,k)\}\ ,$$

$$V_{(i,j)} = \{(a,b,c,d) \in FRT(X) \mid (a,b) = (i,j)\}\ ,$$

$$V_{[i,j]} = \{(a,b,c,d) \in FRT(X) \mid (a,c) = (i,j)\}\ ,$$

$$V_{(i,j]} = \{(a,b,c,d) \in FRT(X) \mid (b,d) = (i,j)\}\ ,$$

$$V_{[i,j)} = \{(a,b,c,d) \in FRT(X) \mid (c,d) = (i,j)\}.$$

The following theorem gives decompositions of $FRT(X)$ into tribands of subtrioids.

Theorem 4. *Let $FRT(X)$ be a free rectangular triband. Then*

(i) $FRT(X)$ is a triband $(X_{lz,rz})^{\dashv}$ of subtrioids $V_{(i)}, i \in (X_{lz,rz})^{\dashv}$, such that $V_{(i)} \cong (FRct(X))^{\vdash}$ for every $i \in X_{lz,rz}$;

(ii) $FRT(X)$ is a triband $(X_{lz,rz})^{\vdash}$ of subtrioids $V_{[i]}, i \in (X_{lz,rz})^{\vdash}$, such that $V_{[i]} \cong (FRct(X))^{\dashv}$ for every $i \in X_{lz,rz}$;

(iii) $FRT(X)$ is a triband $(FRct(X))^{\dashv}$ of subtrioids $V_{(i,j,k)}, (i,j,k) \in (FRct(X))^{\dashv}$, such that $V_{(i,j,k)} \cong (X_{lz,rz})^{\vdash}$ for every $(i,j,k) \in FRct(X)$.

(iv) $FRT(X)$ is a triband $(FRct(X))^{\vdash}$ of subtrioids $V_{[i,j,k]}, (i,j,k) \in (FRct(X))^{\vdash}$, such that $V_{[i,j,k]} \cong (X_{lz,rz})^{\dashv}$ for every $(i,j,k) \in FRct(X)$;

(v) $FRT(X)$ is a triband $(X_{lz,rb})^{\dashv}$ of subtrioids $V_{(i,j)}, (i,j) \in (X_{lz,rb})^{\dashv}$, such that $V_{(i,j)} \cong (X_{rb,rz})^{\vdash}$ for every $(i,j) \in X_{lz,rb}$;

(vi) $FRT(X)$ is a triband $(X_{lz,rb})^{\vdash}$ of subtrioids $V_{[i,j]}, (i,j) \in (X_{lz,rb})^{\vdash}$, such that $V_{[i,j]} \cong (X_{rb,rz})^{\dashv}$ for every $(i,j) \in X_{lz,rb}$;

(vii) $FRT(X)$ is a triband $(X_{rb,rz})^{\dashv}$ of subtrioids $V_{(i,j]}, (i,j) \in (X_{rb,rz})^{\dashv}$, such that $V_{(i,j]} \cong (X_{lz,rb})^{\vdash}$ for every $(i,j) \in X_{rb,rz}$;

(viii) $FRT(X)$ is a triband $(X_{rb,rz})^{\vdash}$ of subtrioids $V_{[i,j)}, (i,j) \in (X_{rb,rz})^{\vdash}$, such that $V_{[i,j)} \cong (X_{lz,rb})^{\dashv}$ for every $(i,j) \in X_{rb,rz}$.

Proof. (i) By Theorem 1 the map

$$\pi^{\dashv}_{lz,rz} : FRT(X) \to (X_{lz,rz})^{\dashv} : (a,b,c,d) \mapsto b$$

is a homomorphism. Then $V_{(i)}, i \in X_{lz,rz}$, is a class of $\Delta_{\pi^{\dashv}_{lz,rz}}$ which is a subtrioid of $FRT(X)$. If $(a_1,b_1,c_1,d_1),(a_2,b_2,c_2,d_2) \in V_{(i)}$, then $b_1 = b_2 = i$ and

$$(a_1,b_1,c_1,d_1) \dashv (a_2,b_2,c_2,d_2) = (a_1,b_1,c_1,d_2) = (a_1,i,c_1,d_2),$$

$$(a_1,b_1,c_1,d_1) \vdash (a_2,b_2,c_2,d_2) = (a_1,b_2,c_2,d_2) = (a_1,i,c_2,d_2),$$

$$(a_1,b_1,c_1,d_1) \perp (a_2,b_2,c_2,d_2) = (a_1,b_1,c_2,d_2) = (a_1,i,c_2,d_2).$$

Hence operations $\vdash$ and $\perp$ of $V_{(i)}$ coincide. It is easy to cheek that for every $i \in X_{lz,rz}$ the map

$$V_{(i)} \to (FRct(X))^{\vdash} : (a,i,c,d) \mapsto (a,c,d)$$

is an isomorphism.

(ii) By Theorem 1 the map

$$\pi^{\vdash}_{lz,rz} : FRT(X) \to (X_{lz,rz})^{\vdash} : (a,b,c,d) \mapsto c$$

is a homomorphism. Then $V_{[i]}, i \in X_{lz,rz}$, is a class of $\Delta_{\pi^{\vdash}_{lz,rz}}$ which is a subtrioid of $FRT(X)$. If $(a_1,b_1,c_1,d_1),(a_2,b_2,c_2,d_2) \in V_{[i]}$, then $c_1 = c_2 = i$ and

$$(a_1,b_1,c_1,d_1) \dashv (a_2,b_2,c_2,d_2) = (a_1,b_1,c_1,d_2) = (a_1,b_1,i,d_2),$$

$$(a_1,b_1,c_1,d_1) \vdash (a_2,b_2,c_2,d_2) = (a_1,b_2,c_2,d_2) = (a_1,b_2,i,d_2),$$

$$(a_1, b_1, c_1, d_1) \bot (a_2, b_2, c_2, d_2) = (a_1, b_1, c_2, d_2) = (a_1, b_1, i, d_2).$$

Hence operations $\dashv$ and $\bot$ of $V_{[i]}$ coincide. It is easy to cheek that for every $i \in X_{lz,rz}$ the map

$$V_{[i]} \to (FRct(X))^{\dashv} : (a, b, i, d) \mapsto (a, b, d)$$

is an isomorphism.

(iii) By Theorem 1 the map

$$\pi^{\dashv}_{FRct} : FRT(X) \to (FRct(X))^{\dashv} : (a, b, c, d) \mapsto (a, b, d)$$

is a homomorphism. Then $V_{(i,j,k)}, (i, j, k) \in FRct(X)$, is a class of $\Delta_{\pi^{\dashv}_{FRct}}$ which is a subtrioid of $FRT(X)$. If $(a_1, b_1, c_1, d_1), (a_2, b_2, c_2, d_2) \in V_{(i,j,k)}$, then $(a_1, b_1, d_1) = (a_2, b_2, d_2) = (i, j, k)$ and

$$(a_1, b_1, c_1, d_1) \dashv (a_2, b_2, c_2, d_2) = (a_1, b_1, c_1, d_2) = (i, j, c_1, k),$$

$$(a_1, b_1, c_1, d_1) \vdash (a_2, b_2, c_2, d_2) = (a_1, b_2, c_2, d_2) = (i, j, c_2, k),$$

$$(a_1, b_1, c_1, d_1) \bot (a_2, b_2, c_2, d_2) = (a_1, b_1, c_2, d_2) = (i, j, c_2, k).$$

Hence operations $\vdash$ and $\bot$ of $V_{(i,j,k)}$ coincide. It is clear that for every $(i, j, k) \in FRct(X)$ the map

$$V_{(i,j,k)} \to (X_{lz,rz})^{\vdash} : (i, j, c, k) \mapsto c$$

is an isomorphism.

(iv) By Theorem 1 the map

$$\pi^{\vdash}_{FRct} : FRT(X) \to (FRct(X))^{\vdash} : (a, b, c, d) \mapsto (a, c, d)$$

is a homomorphism. Then $V_{[i,j,k]}, (i, j, k) \in FRct(X)$, is a class of $\Delta_{\pi^{\vdash}_{FRct}}$ which is a subtrioid of $FRT(X)$. If $(a_1, b_1, c_1, d_1), (a_2, b_2, c_2, d_2) \in V_{[i,j,k]}$, then $(a_1, c_1, d_1) = (a_2, c_2, d_2) = (i, j, k)$ and

$$(a_1, b_1, c_1, d_1) \dashv (a_2, b_2, c_2, d_2) = (a_1, b_1, c_1, d_2) = (i, b_1, j, k),$$

$$(a_1, b_1, c_1, d_1) \vdash (a_2, b_2, c_2, d_2) = (a_1, b_2, c_2, d_2) = (i, b_2, j, k),$$

$$(a_1, b_1, c_1, d_1) \bot (a_2, b_2, c_2, d_2) = (a_1, b_1, c_2, d_2) = (i, b_1, j, k).$$

Hence operations $\dashv$ and $\bot$ of $V_{[i,j,k]}$ coincide. One can verify that for every $(i, j, k) \in FRct(X)$ the map

$$V_{[i,j,k]} \to (X_{lz,rz})^{\dashv} : (i, b, j, k) \mapsto b$$

is an isomorphism.

(v) By Theorem 1 the map

$$\pi^{\dashv}_{lz,rb} : FRT(X) \to (X_{lz,rb})^{\dashv} : (a, b, c, d) \mapsto (a, b)$$

is a homomorphism. Then $V_{(i,j)}, (i,j) \in X_{lz,rb}$, is a class of $\Delta_{\pi^{\dashv}_{lz,rb}}$ which is a subtrioid of $FRT(X)$. If $(a_1, b_1, c_1, d_1), (a_2, b_2, c_2, d_2) \in V_{(i,j)}$, then $(a_1, b_1) = (a_2, b_2) = (i, j)$ and

$$(a_1, b_1, c_1, d_1) \dashv (a_2, b_2, c_2, d_2) = (a_1, b_1, c_1, d_2) = (i, j, c_1, d_2),$$

$$(a_1, b_1, c_1, d_1) \vdash (a_2, b_2, c_2, d_2) = (a_1, b_2, c_2, d_2) = (i, j, c_2, d_2),$$

$$(a_1, b_1, c_1, d_1) \perp (a_2, b_2, c_2, d_2) = (a_1, b_1, c_2, d_2) = (i, j, c_2, d_2).$$

Hence operations $\vdash$ and $\perp$ of $V_{(i,j)}$ coincide. One can cheek that for every $(i, j) \in X_{lz,rb}$ the map

$$V_{(i,j)} \to (X_{rb,rz})^{\vdash} : (i, j, c, d) \mapsto (c, d)$$

is an isomorphism.

(vi) By Theorem 1 the map

$$\pi^{\vdash}_{lz,rb} : FRT\ (X) \to (X_{lz,rb})^{\vdash} : (a, b, c, d) \mapsto (a, c)$$

is a homomorphism. Then $V_{[i,j]}, (i, j) \in X_{lz,rb}$, is a class of $\Delta_{\pi^{\vdash}_{lz,rb}}$ which is a subtrioid of $FRT(X)$. If $(a_1, b_1, c_1, d_1), (a_2, b_2, c_2, d_2) \in V_{[i,j]}$, then $(a_1, c_1) = (a_2, c_2) = (i, j)$ and

$$(a_1, b_1, c_1, d_1) \dashv (a_2, b_2, c_2, d_2) = (a_1, b_1, c_1, d_2) = (i, b_1, j, d_2),$$

$$(a_1, b_1, c_1, d_1) \vdash (a_2, b_2, c_2, d_2) = (a_1, b_2, c_2, d_2) = (i, b_2, j, d_2),$$

$$(a_1, b_1, c_1, d_1) \perp (a_2, b_2, c_2, d_2) = (a_1, b_1, c_2, d_2) = (i, b_1, j, d_2).$$

Hence operations $\dashv$ and $\perp$ of $V_{[i,j]}$ coincide. It can be shown that for every $(i, j) \in X_{lz,rb}$ the map

$$V_{[i,j]} \to (X_{rb,rz})^{\dashv} : (i, b, j, d) \mapsto (b, d)$$

is an isomorphism.

(vii) By Theorem 1 the map

$$\pi^{\dashv}_{rb,rz} : FRT\ (X) \to (X_{rb,rz})^{\dashv} : (a, b, c, d) \mapsto (b, d)$$

is a homomorphism. Then $V_{(i,j]}, (i, j) \in X_{rb,rz}$, is a class of $\Delta_{\pi^{\dashv}_{rb,rz}}$ which is a subtrioid of $FRT(X)$. If $(a_1, b_1, c_1, d_1), (a_2, b_2, c_2, d_2) \in V_{(i,j]}$, then $(b_1, d_1) = (b_2, d_2) = (i, j)$ and

$$(a_1, b_1, c_1, d_1) \dashv (a_2, b_2, c_2, d_2) = (a_1, b_1, c_1, d_2) = (a_1, i, c_1, j),$$

$$(a_1, b_1, c_1, d_1) \vdash (a_2, b_2, c_2, d_2) = (a_1, b_2, c_2, d_2) = (a_1, i, c_2, j),$$

$$(a_1, b_1, c_1, d_1) \perp (a_2, b_2, c_2, d_2) = (a_1, b_1, c_2, d_2) = (a_1, i, c_2, j).$$

Hence operations $\vdash$ and $\perp$ of $V_{(i,j]}$ coincide. Clearly, for every $(i, j) \in X_{rb,rz}$ the map

$$V_{(i,j]} \to (X_{lz,rb})^{\vdash} : (a, i, c, j) \mapsto (a, c)$$

is an isomorphism.

(viii) By Theorem 1 the map

$$\pi^{\vdash}_{rb,rz} : FRT(X) \to (X_{rb,rz})^{\vdash} : (a,b,c,d) \mapsto (c,d)$$

is a homomorphism. Then $V_{[i,j)}, (i,j) \in X_{rb,rz}$, is a class of $\Delta_{\pi^{\vdash}_{rb,rz}}$ which is a subtrioid of $FRT(X)$. If $(a_1,b_1,c_1,d_1),(a_2,b_2,c_2,d_2) \in V_{[i,j)}$, then $(c_1,d_1) = (c_2,d_2) = (i,j)$ and

$$(a_1,b_1,c_1,d_1)\dashv(a_2,b_2,c_2,d_2) = (a_1,b_1,c_1,d_2) = (a_1,b_1,i,j),$$

$$(a_1,b_1,c_1,d_1)\vdash(a_2,b_2,c_2,d_2) = (a_1,b_2,c_2,d_2) = (a_1,b_2,i,j),$$

$$(a_1,b_1,c_1,d_1)\perp(a_2,b_2,c_2,d_2) = (a_1,b_1,c_2,d_2) = (a_1,b_1,i,j).$$

Hence operations $\dashv$ and $\perp$ of $V_{[i,j)}$ coincide. Evidently, for every $(i,j) \in X_{rb,rz}$ the map

$$V_{[i,j)} \to (X_{lz,rb})^{\dashv} : (a,b,i,j) \mapsto (a,b)$$

is an isomorphism. □

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Yuliia V. Zhuchok
Department of Algebra and System Analysis
Luhansk Taras Shevchenko National University
Gogol square, 1, Starobilsk, 92700, Ukraine
E-mail: *yulia.mih@mail.ru*

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