The endotopism spectrum of an equivalence

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Let X be an arbitrary nonempty set and $\rho \subseteq X \times X$. The ordered pair (φ, ψ) of transformations φ and ψ of a set X is called an *endotopism* [1] of ρ if for all $a, b \in X$ the condition $(a, b) \in \rho$ implies $(a\varphi, b\psi) \in \rho$. The set of all endotopisms of ρ is denoted by $Et(\rho)$.

The endotopism $(\varphi, \psi) \in Et(\rho)$ is called a *half-strong endotopism* if for all $a, b \in X$ the condition $(a\varphi, b\psi) \in \rho$ implies the existence of preimages $a' \in a\varphi\varphi^{-1}$ and $b' \in b\psi\psi^{-1}$ such that $(a', b') \in \rho$. We denote the set of all half-strong endotopisms of ρ by $HEt(\rho)$.

The endotopism $(\varphi, \psi) \in Et(\rho)$ is called a *locally strong endotopism* if for all $a, b \in X$ the condition $(a\varphi, b\psi) \in \rho$ implies that for every $a' \in a\varphi\varphi^{-1}$ there exists $b' \in b\psi\psi^{-1}$ such that $(a', b') \in \rho$ and analogously for all preimages of $b\psi$. By $LEt(\rho)$ we denote the set of all locally strong endotopisms of ρ.

The endotopism $(\varphi, \psi) \in Et(\rho)$ is called a *quasi-strong endotopism* if for all $a, b \in X$ the condition $(a\varphi, b\psi) \in \rho$ implies that there exists $a' \in a\varphi\varphi^{-1}$ such that for every $b' \in b\psi\psi^{-1}$ we have $(a', b') \in \rho$ and analogously for a suitable preimage of $b\psi$. We denote the set of all quasi-strong endotopisms of ρ by $QEt(\rho)$.

The endotopism $(\varphi, \psi) \in Et(\rho)$ is called a *strong endotopism* if for all $a, b \in X$ the condition $(a\varphi, b\psi) \in \rho$ implies $(a, b) \in \rho$. By $SEt(\rho)$ we denote the set of all strong endotopisms of ρ.

The ordered pair (φ, ψ) of permutations φ and ψ of a set X is called an *autotopism* of $\rho \subseteq X \times X$ if $(a, b) \in \rho$ iff $(a\varphi, b\psi) \in \rho$ for all $a, b \in X$. The set of all autotopisms of ρ we denote by $At(\rho)$.

Note that for any $\rho \subseteq X \times X$, we have the following sequence of inclusions: $Et(\rho) \supseteq HEt(\rho) \supseteq LEt(\rho) \supseteq QEt(\rho) \supseteq SEt(\rho) \supseteq At(\rho)$. With this sequence we associate the respective cardinalities: $Etspec(X, \rho) = (|Et(\rho)|, |HEt(\rho)|, |LEt(\rho)|, |QEt(\rho)|, |SEt(\rho)|, |At(\rho)|)$, which is called the *endospectrum* (or the *endotopism spectrum*) of ρ relative to its endotopisms.

We denote the set of all equivalence relations on a set X by $Eq(X)$. Let $\alpha \in Eq(X)$, $\Im(X)$ be the symmetric semigroup on X and $S(X)$ be the symmetric group on X. By $B(X/\alpha)$ we denote the set of all bijections $\eta : X/\alpha \to X/\alpha$ such that $|A| = |A\eta|$ for all $A \in X/\alpha$.

Theorem. *For any equivalence α on a finite set X, we have:*

(i) $|Et(\alpha)| = \sum_{\tau^* \in \Im(X/\alpha)} (\prod_{A \in X/\alpha} |A\tau^*|^{|A|})^2$;

(ii) $|LEt(\alpha)| = \sum_{\tau^* \in \Im(X/\alpha)} (\prod_{A \in (X/\alpha)\tau^*} (|A| + \sum_{i=2}^{|M|} C_{|A|}^i \prod_{A' \in A\tau^{*-1}} (i^{|A'|} - C_i^{i-1}((i-1)^{|A'|} - (i-1)) - i)))^2$, *where* $|M| \geq 2$ *and* $M \in A\tau_{min}^{*-1} = \{A' \in A\tau^{*-1} \cup \{A\} : \mid |A'| \leq |A|\}$ *be the class of a least cardinality;*

(iii) $|SEt(\alpha)| = \sum_{\tau^* \in S(X/\alpha)} (\prod_{A \in X/\alpha} |A\tau^*|^{|A|})^2$*;*

(iv) $|QEt(\alpha)| = |SEt(\alpha)|$;

(v) $|At(\alpha)| = |B(X/\alpha)| \cdot (\prod_{A \in X/\alpha} |A|!)^2$.

A question of calculating $|HEt(\alpha)|$ for any equivalence $\alpha \in Eq(X)$ is open.

1. Popov B. V. Semigroups of endomorphisms of μ-ary relations. Uchen. Zapiski LGPI im. A.I. Gertsena, 1965, **274**, P. 184-201 (In Russian).
2. Clifford A.H., Preston G.B. The algebraic theory of semigroups, V.1. – M.: Mir, 1972, 506 p. (In Russian).