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On the saturations of submodules

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Abstract. Let $R \subseteq S$ be a ring extension, and let A be an R-submodule of S. The saturation of A (in S) by τ is set $A_{[\tau]} = \{x \in S : tx \in A \text{ for some } t \in \tau\}$, where τ is a multiplicative subset of R. We study properties of saturations of R-submodules of S. We use this notion of saturation to characterize star operations $\star$ on ring extensions $R \subseteq S$ satisfying the relation $(A \cap B)^\star = A^\star \cap B^\star$ whenever A and B are two R-submodules of S such that $AS = BS = S$.

1. Introduction

Throughout this paper, all rings considered are commutative with identity. Let $R \subseteq S$ be a ring extension, and let A be an R-submodule of S. The *saturation* of A (in S) by τ is set

$$A_{[\tau]} = \{x \in S : tx \in A \text{ for some } t \in \tau\},$$

where τ is a multiplicative subset of R [7, Definition 10, p. 18]. If $\mathfrak{p}$ is a prime ideal of R, then the saturation of A with respect to $\tau = R \setminus \mathfrak{p}$ is denoted $A_{[\mathfrak{p}]}$. If A is an R-submodule A of S and $\mathcal{M}$ is the set of all maximal ideals of R, then $A = \bigcap_{\mathfrak{p}\in\mathcal{M}}(R_{[\mathfrak{p}]}A_{[\mathfrak{p}]})$ and $R = \bigcap_{\mathfrak{p}\in\mathcal{M}} R_{[\mathfrak{p}]}$ [7, Remark 5.5, p. 50]. In their book [7], Knebusch and Zhang use this

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notion of saturation to define and study Prüfer extension; see [7, pages 46-73]. In her paper [8], McNair studies properties of saturations of the form $I_\tau = \{x \in S : tx \in I \text{ for some } t \in \tau\}$, when I is an ideal of R, and τ is a multiplicative subset of R. In particular, McNair proved that if $R \subseteq S$ is a ring extension, and I is a radical ideal of R, then $I_{[\mathfrak{p}]} \cap R = \mathfrak{p}$ and $I_{[\mathfrak{p}]} = \mathfrak{p}_{[\mathfrak{p}]}$ for any minimal prime ideal $\mathfrak{p}$ of I [8, Proposition 2.1]. Furthermore, if $R \subseteq S \subseteq T$ and $\mathfrak{p}$ is a prime ideal of S such that $S_{[\mathfrak{p}]} = R_{[\mathfrak{p} \cap R]}$ (saturation in T), then $S_{\mathfrak{p}} = R_{\mathfrak{p} \cap R}$ [8, Lemma 2.7]. In the present paper, we do not limit our study to only ideals of R but to any R-submodule of S.

In Section 2, we study properties of saturations, and we show that the saturation is distributive with respect to the finite intersection (and finite sum). Note that when R is a domain with quotient field S, the notion of saturation coincides with the notion of localization, and $AR_{[\mathfrak{p}]} = A_{[\mathfrak{p}]}$ for any R-submodule A of S. In Example 2.5, we construct a ring extension $R \subseteq S$ and an R-submodule A of S such that $AR_{[\mathfrak{p}]} \subsetneq A_{[\mathfrak{p}]}$. In Proposition 2.7, Proposition 2.9 and Proposition 2.11, we investigate conditions under which the equality $AR_{[\mathfrak{p}]} = A_{[\mathfrak{p}]}$ is satisfied for a ring extension $R \subseteq S$ and an R-submodule A of S.

In Section 3, we use the notion of saturation to study properties of star operations on ring extensions. In Theorem 3.10, which is a generalization of [2, Theorem 4] due to D.D Anderson, we characterize star operations $\star$ on a ring extension $R \subseteq S$ satisfying the relation $(A \cap B)^\star = A^\star \cap B^\star$, where A and B are S-regular R-submodules of S.

2. Some properties of saturations of submodules

In this section, we consider a ring extension $R \subseteq S$ and we study properties of the saturations of R-submodules of S. Let A be an R-submodule of S. The *saturation* of A (in S) by τ is set $A_{[\tau]} = \{x \in S : tx \in A \text{ for some } t \in \tau\}$, where τ is a multiplicative subset of R [7, Definition 10, p. 18]. If $\mathfrak{p}$ is a prime ideal of R, then the saturation of A with respect to $\tau = R \setminus \mathfrak{p}$ is denoted $A_{[\mathfrak{p}]}$. For each R-submodule A of S, the set $[R :_S A]$ denotes the set of all $x \in S$ such that $xA \subseteq R$, and $(R :_A A)$ denotes the set of all $x \in R$ such that $xA \subseteq R$ [7, Definition 2, p. 85]. We also use the notations $[R : A]$ and $(R : A)$ when the context is clear. The R-submodule A of S is called S-*invertible*, if there exists an R-submodule B of S such that $AB = R$ [7, Definition 3, p. 90]. In this case, we write $B = A^{-1}$, and $A^{-1} = [R :_S A] = \{x \in S : xA \subseteq R\}$ [7, Remarks 1.10, p. 90].

Proposition 2.1. *Let $R \subseteq S$ be a ring extension, and let τ be a multiplicative subset of R. Let A and B be two R-submodules of S. Then*
(1) $AR_{[\tau]} \subseteq A_{[\tau]}$.
(2) $(A \cap B)_{[\tau]} = A_{[\tau]} \cap B_{[\tau]}$.
(3) $(A + B)_{[\tau]} = (A_{[\tau]} + B_{[\tau]})_{[\tau]}$.
(4) *If B is finitely generated, then* $(A_{[\tau]} : B_{[\tau]}) = (A : B)_{[\tau]}$.

Proof. (1) Let $x \in AR_{[\tau]}$. Then $x = \sum_{i=1}^{\ell} a_i u_i$ with $a_i \in A$ and $u_i \in R_{[\tau]}$ for $1 \leqslant i \leqslant \ell$. Let $t_i \in \tau$ such that $t_i u_i \in R$ for each $1 \leqslant i \leqslant \ell$, and let $t = \prod_{i=1}^{\ell} t_i$. Then $tx = \sum_{i=1}^{\ell} a_i(tu_i) \in A$. It follows that $x \in A_{[\tau]}$. This shows that $AR_{[\tau]} \subseteq A_{[\tau]}$.

(2) Let $x \in (A \cap B)_{[\tau]}$. Then there exists $t \in \tau$ such that $tx \in A \cap B$. It follows that $tx \in A$ and $tx \in B$. Therefore, $x \in A_{[\tau]}$ and $x \in B_{[\tau]}$. Thus $x \in A_{[\tau]} \cap B_{[\tau]}$. This shows that $(A \cap B)_{[\tau]} \subseteq A_{[\tau]} \cap B_{[\tau]}$. On the other hand, if x is an element of $A_{[\tau]} \cap B_{[\tau]}$, then there exist $t_1, t_2 \in \tau$ such that $t_1 x \in A$ and $t_2 x \in B$. It follows that $t_1 t_2 x \in A \cap B$. So $x \in (A \cap B)_{[\tau]}$, since $t_1 t_2 \in \tau$. This shows that $A_{[\tau]} \cap B_{[\tau]} \subseteq (A \cap B)_{[\tau]}$. Therefore, $(A \cap B)_{[\tau]} = A_{[\tau]} \cap B_{[\tau]}$.

(3) By the definition, the containment $A \subseteq A_{[\tau]}$ is always true. Thus $(A + B)_{[\tau]} \subseteq (A_{[\tau]} + B_{[\tau]})_{[\tau]}$, since $A + B \subseteq A_{[\tau]} + B_{[\tau]}$. Now let $x \in (A_{[\tau]} + B_{[\tau]})_{[\tau]}$. Then there exists $t \in \tau$ such that $tx = x_1 + x_2$ for some $x_1 \in A_{[\tau]}$ and $x_2 \in B_{[\tau]}$. Let $t_1, t_2 \in \tau$ such that $t_1 x_1 \in A$ and $t_2 x_2 \in B$. Then $tt_1 t_2 x \in A + B$. It follows that $x \in (A + B)_{[\tau]}$, since $tt_1 t_2 \in \tau$. This shows that $(A_{[\tau]} + B_{[\tau]})_{[\tau]} \subseteq (A + B)_{[\tau]}$. Therefore, $(A + B)_{[\tau]} = (A_{[\tau]} + B_{[\tau]})_{[\tau]}$.

(4) Let $u \in (A : B)_{[\tau]}$. Then $tu \in (A : B)$ for some $t \in \tau$. It follows that $tuB \subseteq A$. Let $x \in B_{[\tau]}$. Then there exists $t' \in \tau$ such that $t'x \in B$. Thus $tt'xu = (tu)(t'x) \in A$. Hence $xu \in A_{[\tau]}$. It follows that $u \in (A_{[\tau]} : B_{[\tau]})$, since x was chosen arbitrarily in $B_{[\tau]}$. This shows that $(A : B)_{[\tau]} \subseteq (A_{[\tau]} : B_{[\tau]})$. It remains to show that $(A_{[\tau]} : B_{[\tau]}) \subseteq (A : B)_{[\tau]}$. Let $y_1, \ldots, y_n \in S$ such that $B = (y_1, \ldots, y_n)R$.

Step 1. If $n = 1$, then B is generated by the element y_1 of S. Let $x \in (A_{[\tau]} : B_{[\tau]})$. Then $xB_{[\tau]} \subseteq A_{[\tau]}$. It follows that $xy_1 \in A_{[\tau]}$, since $y_1 \in B \subseteq B_{[\tau]}$. So there exists $t \in \tau$ such that $txy_1 = a$ for some $a \in A$. Let $b \in B$. Then $b = ry_1$ for some $r \in R$. Therefore, $t(xb) = r(txy_1) = ra \in A$. It follows that $tx \in (A : B)$, since b was chosen arbitrarily in B. Hence $x \in (A : B)_{[\tau]}$. This shows that $(A_{[\tau]} : B_{[\tau]}) \subseteq (A : B)_{[\tau]}$.

Step 2. More generally, for $n \geqslant 1$, we have $B = B_1 + \cdots + B_n$, where each B_i is the R-submodule of S generated by y_i, $1 \leqslant i \leqslant n$. So

$$(A : B)_{[\tau]} = (A : B_1 + \cdots + B_n)_{[\tau]} = (\bigcap_{i=1}^{n} (A : B_i))_{[\tau]}.$$

But by (2), we have $(\bigcap_{i=1}^{n}(A:B_i))_{[\tau]} = \bigcap_{i=1}^{n}(A:B_i)_{[\tau]}$. Furthermore, by Step 1, we have $\bigcap_{i=1}^{n}(A:B_i)_{[\tau]} = \bigcap_{i=1}^{n}(A_{[\tau]}:B_{i[\tau]})$. So

$$(A:B)_{[\tau]} = \bigcap_{i=1}^{n}(A_{[\tau]}:B_{i[\tau]}) = (A_{[\tau]}:B_{1[\tau]}+\cdots+B_{n[\tau]}).$$

But by (3), we have

$$B_{1[\tau]}+\cdots+B_{n[\tau]} \subseteq (B_{1[\tau]}+\cdots+B_{n[\tau]})_{[\tau]} = (B_1+\cdots+B_n)_{[\tau]} = B_{[\tau]}.$$

It follows that $(A_{[\tau]}:B_{[\tau]}) \subseteq (A_{[\tau]}:B_{1[\tau]}+\cdots+B_{n[\tau]}) = (A:B)_{[\tau]}$. $\square$

Lemma 2.2. *Let $R \subseteq S$ be a ring extension, and suppose that $\mathfrak{p}$ is a prime ideal of R. Then $\mathfrak{p} = \mathfrak{p}R_{[\mathfrak{p}]} \cap R = \mathfrak{p}_{[\mathfrak{p}]} \cap R$.*

Proof. The relation $\mathfrak{p} \subseteq \mathfrak{p}R_{[\mathfrak{p}]} \cap R \subseteq \mathfrak{p}_{[\mathfrak{p}]} \cap R$ is a direct consequence of Proposition 2.1(1). Now we show that $\mathfrak{p}_{[\mathfrak{p}]} \cap R \subseteq \mathfrak{p}$. Let $x \in \mathfrak{p}_{[\mathfrak{p}]} \cap R$. Then there exists $t \in R \setminus \mathfrak{p}$ such that $tx \in \mathfrak{p}$. It follows that $x \in \mathfrak{p}$, since $\mathfrak{p}$ is a prime ideal of R and $t \notin \mathfrak{p}$. This shows that $\mathfrak{p}_{[\mathfrak{p}]} \cap R \subseteq \mathfrak{p}$. Thus $\mathfrak{p} = \mathfrak{p}R_{[\mathfrak{p}]} \cap R = \mathfrak{p}_{[\mathfrak{p}]} \cap R$. $\square$

Remark 2.3. In [8, Proposition 2.1], McNair showed that if $R \subseteq S$ is a ring extension and I is a radical ideal of R, then $I_{[\mathfrak{p}]} \cap R = \mathfrak{p}$ for any minimal prime ideal $\mathfrak{p}$ of I. So by taking $I = \mathfrak{p}$, we obtain Lemma 2.2.

For the rest of the article, if R is a ring, we denote by $\mathrm{Spec}(R)$ the set of all prime ideals of R. The intersection of all prime ideal of R be denoted by $\mathrm{Nil}(R)$. A ring extension is said to be *tight* if for every $x \in S \setminus R$, there exists an S-invertible ideal I of R such that $xI \subseteq R$ [7, Definition 1, p. 94].

Proposition 2.4. *Let $R \subseteq S$ be a tight ring extension. If $\mathrm{Nil}(R) = 0$ then,*

$$\bigcup_{\mathfrak{p}\in \mathrm{Spec}(R)} R_{[\mathfrak{p}]} = S.$$

Proof. Let $x \in S$, then there exists an S-invertible ideal I of R such that $xI \subseteq R$. Furthermore, we have $IS = S$. Thus $I \neq 0$. It follows that there exists $\mathfrak{p} \in \mathrm{Spec}(R)$ such that $I \nsubseteq \mathfrak{p}$. Let $t \in I \setminus \mathfrak{p}$. Then $tx \in R$. Hence $x \in R_{[\mathfrak{p}]}$. This shows that $S \subseteq \cup_{\mathfrak{p}\in\mathrm{Spec}(R)} R_{[\mathfrak{p}]}$. The containment $\cup_{\mathfrak{p}\in\mathrm{Spec}(R)} R_{[\mathfrak{p}]} \subseteq S$ is clear. Thus $S = \bigcup_{\mathfrak{p}\in\mathrm{Spec}(R)} R_{[\mathfrak{p}]}$. $\square$

In the next example, we construct a ring extension $R \subseteq S$, a prime ideal $\mathfrak{p}$ of R and an R-submodule A of S such that $A_{[\mathfrak{p}]} \neq AR_{[\mathfrak{p}]}$.

Example 2.5. Let D be an integral domain, and let X, Y be two indeterminates over D. Let $R = D[X^2, Y]$ and $S = D[X, Y]$. Let $\mathfrak{p}$ be the ideal of R generated by Y, and let A be the R-submodule of S generated by X^2Y. Then $A_{[\mathfrak{p}]} \neq AR_{[\mathfrak{p}]}$.

Proof. Let $t = X^4 \in R$. Then $t \notin \mathfrak{p}$ and $tY = YX^4 = (X^2Y)X^2 \in A$. This shows that $Y \in A_{[\mathfrak{p}]}$. Now we show that $Y \notin AR_{[\mathfrak{p}]}$. By contradiction, suppose that $Y \in AR_{[\mathfrak{p}]}$. Then $Y = \sum_{i=1}^{\ell} X^2YU_iV_i$ with $U_i \in R$ and $V_i \in R_{[\mathfrak{p}]}$ for $1 \leqslant i \leqslant \ell$. It follows that $1 = \sum_{i=1}^{\ell} X^2U_iV_i$. This last equality is not possible, since X is not a unit in S. Thus $Y \notin AR_{[\mathfrak{p}]}$. Therefore, $A_{[\mathfrak{p}]} \neq AR_{[\mathfrak{p}]}$. □

For the rest of this section, we investigate conditions under which $A_{[\mathfrak{p}]} = AR_{[\mathfrak{p}]}$, when A is an R-submodule of S and $\mathfrak{p}$ is a prime ideal of R. First we recall some definitions. Let T be a ring, and let Γ be an additive totally ordered abelian group. Let $\Gamma \cup \infty = \Gamma \cup \{\infty\}$, where $\infty + g = g + \infty = \infty$ for all $g \in \Gamma \cup \infty$, and $g < \infty$ for all $g \in \Gamma$. A *valuation* on T with values in Γ is a map $v : T \to \Gamma \cup \infty$ such that:

(1) $v(xy) = v(x) + v(y)$ for all $x, y \in T$.
(2) $v(x + y) \geqslant \min\{v(x), v(y)\}$ for all $x, y \in T$.
(3) $v(1) = 0$ and $v(0) = \infty$.

If $v(T) = \Gamma \cup \infty$, then v is called a *Manis valuation* on S [7, Definition 4, p. 12]. In this case, $V = \{x \in T : v(x) \geqslant 0\}$ is called a *Manis subring* of T, the extension $V \subseteq T$ is called a *Manis extension*, and $(V, \mathfrak{p})$ is called a *Manis pair* in T, where $\mathfrak{p} = \{x \in T : v(x) > 0\}$ [7, Definition 1, p. 22].

Let $R \subseteq S$ be a ring extension, and let A be an R-submodule of S. The R-submodule A is said to be *S-regular* if $AS = S$ [7, Definition 1, p. 84]. The ring S is called a *Prüfer extension* of R if $(R_{[\mathfrak{p}]}, \mathfrak{p}_{[\mathfrak{p}]})$ is a Manis pair in S for every maximal ideal $\mathfrak{p}$ of R. In this case, we say that R is Prüfer subring of S. The ring extension $R \subseteq S$ is called *Bézout extension* if $R \subseteq S$ is a Prüfer extension and every S-invertible ideal of R is a principal ideal. More on Manis valuations, Prüfer extensions and Bézout extensions can be found in [7].

Lemma 2.6 ([7, Theorem 1.13, p. 91])**.** *If $R \subseteq S$ is a Prüfer extension, then every finitely generated S-regular R-submodule of S is S-invertible.*

Proposition 2.7. *Let $R \subseteq S$ be a ring extension, and let A be an R-submodule of S. If A is S-invertible, then for any prime ideal $\mathfrak{p}$ of R, $AR_{[\mathfrak{p}]} = A_{[\mathfrak{p}]}$.*

Proof. Let A be an S-invertible R-submodule of S. By Proposition 2.1(1), we have $AR_{[\mathfrak{p}]} \subseteq A_{[\mathfrak{p}]}$. Thus it suffices to show that $A_{[\mathfrak{p}]} \subseteq AR_{[\mathfrak{p}]}$. Let $x \in A_{[\mathfrak{p}]}$. Then there exists $t \in R \setminus \mathfrak{p}$ such that $tx \in A$. Since $AA^{-1} = R$, there exist $a_1, \dots, a_\ell \in A$ and $b_1, \dots, b_\ell \in A^{-1}$ such that $a_1b_1 + \cdots + a_\ell b_\ell = 1$. Moreover, for $1 \leqslant i \leqslant \ell$, we have $txb_i \in R$, since $tx \in A$ and $b_i \in A^{-1}$. Thus $xb_i \in R_{[\mathfrak{p}]}$ for $1 \leqslant i \leqslant \ell$. It follows that $x = \sum_{i=0}^{\ell} a_i(xb_i) \in AR_{[\mathfrak{p}]}$. □

Remark 2.8. Let $R \subseteq S$ be a Prüfer extension, and let A be an S-regular finitely generated R-submodule of S. Then by Lemma 2.6, A is S-invertible. It follows from Proposition 2.7 that $AR_{[\mathfrak{p}]} = A_{[\mathfrak{p}]}$ for any prime ideal $\mathfrak{p}$ of R.

We recall from [3, Definition 2, p.13] that and R-module M is flat, if for every injective homomorphism $v : M_1 \to M_2$ of R-modules, the R-module homomorphism $1_M \otimes v : M \otimes_R M_1 \to M \otimes_R M_2$ is also injective.

Theorem 2.9. *Let $R \subseteq S$ be a ring extension, and suppose that $R = \bigcap_{i=1}^{n} V_i$. If the extension $V_i \subseteq S$ is Prüfer for each $1 \leqslant i \leqslant n$, then $AR_{[\mathfrak{p}]} = A_{[\mathfrak{p}]}$ for any S-regular flat R-submodule A of S and any prime ideal $\mathfrak{p}$ of R.*

Proof. Let A be an S-regular flat R-submodule of S. Since $AS = S$, there exist $u_1, \dots, u_\ell \in A$ and $s_1, \dots, s_\ell \in S$ such that $u_1s_1 + \cdots + u_\ell s_\ell = 1$. Let A_i be the V_i-submodule of S generated by $u_1, \dots, u_\ell$. Then it is clear that $A_iS = S$. Hence A_i is an S-regular finitely generated V_i-submodule of S.

Let $x \in A_{[\mathfrak{p}]}$. Then there exists $t \in R \setminus \mathfrak{p}$ such that $tx \in A$. Let $B_i = txV_i + A_i$. Then B_i is also an S-regular finitely generated V_i-submodule of S. It follows from Lemma 2.6 that B_i is an S-invertible V_i-submodule of S. Therefore, there exists a V_i-submodule C_i of S such that $B_iC_i = V_i$. Let $c_{i0}, c_{i1}, \dots, c_{i\ell} \in C_i$ such that $c_{i0}tx + c_{i1}u_1 + \cdots + c_{i\ell}u_\ell = 1$. Then $x = (tx)(xc_{i0}) + u_1(xc_{i1}) + \cdots + u_\ell(xc_{i\ell})$. Now notice that $t(c_{ik}x) = (tx)c_{ik} \in B_iC_i = V_i$ for $0 \leqslant k \leqslant \ell$ and $1 \leqslant i \leqslant \ell$. It follows that $c_{ik}x \in V_{i[\mathfrak{p}]}$ for $0 \leqslant k \leqslant \ell$ and $1 \leqslant i \leqslant \ell$. Therefore, for $1 \leqslant i \leqslant \ell$, we have $x \in AV_{i[\mathfrak{p}]}$, since $tx, u_1, \dots, u_\ell \in A$. This shows that $x \in \bigcap_{i=1}^{n} AV_{i[\mathfrak{p}]}$. Since A is a flat R-module, we have $\bigcap_{i=1}^{n} AV_{i[\mathfrak{p}]} = A(\bigcap_{i=1}^{n} V_{i[\mathfrak{p}]})$ [3, Proposition 6, p. 18]. Furthermore, by Proposition 2.1(2), we have $\bigcap_{i=1}^{n} V_{i[\mathfrak{p}]} = (\bigcap_{i=1}^{n} V_i)_{[\mathfrak{p}]} = R_{[\mathfrak{p}]}$. It follows that $x \in AR_{[\mathfrak{p}]}$. This shows that $A_{[\mathfrak{p}]} \subseteq AR_{[\mathfrak{p}]}$, and hence $A_{[\mathfrak{p}]} = AR_{[\mathfrak{p}]}$. □

Remark 2.10. Let $R \subseteq S$ be a Prüfer extension, and let A be an S-regular R-submodule of S.

(1) If A is finitely generated, then $A_{[\mathfrak{p}]} = AR_{[\mathfrak{p}]}$; see Remark 2.8.

(2) If A is a flat R-submodule of S, then $A_{[\mathfrak{p}]} = AR_{[\mathfrak{p}]}$. This is obtained by taking $n = 1$ in the hypothesis of Proposition 2.9.

In the next result, we show that in a Prüfer extension $R \subseteq S$, the equality $AR_{[\mathfrak{p}]} = A_{[\mathfrak{p}]}$ is always true for any S-regular R-submodule A of S and any prime ideal $\mathfrak{p}$ of R (no other condition is needed). The proof of Proposition 2.11 is a modification of Proposition 2.9.

Proposition 2.11. *If $R \subseteq S$ is a Prüfer extension, then for any S-regular R-submodule of S, we have $AR_{[\mathfrak{p}]} = A_{[\mathfrak{p}]}$ for each prime ideal $\mathfrak{p}$ of R.*

Proof. Let A be an S-regular R-submodule of S, and let $x \in A_{[\mathfrak{p}]}$. Then there exists $t \in R \setminus \mathfrak{p}$ such that $tx \in A$. Since $AS = S$, there exist $a_1, \ldots, a_\ell \in A$ and $s_1, \ldots, s_\ell \in S$ such that $a_1s_1 + \cdots + a_\ell s_\ell = 1$. Let A' be the R-submodule of S generated by $a_1, \cdots, a_\ell$. Then it is clear that A' is an S-regular finitely generated R-submodule of S. Let $B = txR + A'$. Then B is also an S-regular finitely generated R-submodule of S. It follows from Lemma 2.6 that B is S-invertible. Hence, there exist $b_0, b_1, \ldots, b_\ell \in B^{-1}$ such that $b_0tx + b_1a_1 + \cdots + b_\ell a_\ell = 1$. But for $0 \leqslant i \leqslant \ell$, we have $t(b_ix) = b_i(tx) \in R$, since $tx \in B$ and $b_i \in B^{-1}$. It follows that $b_ix \in R_{[\mathfrak{p}]}$ for each $0 \leqslant i \leqslant \ell$. Therefore, $x = (tx)(xb_0) + a_1(xb_1) + \cdots + a_\ell(xb_\ell) \in AR_{[\mathfrak{p}]}$. This shows that $A_{[\mathfrak{p}]} \subseteq AR_{[\mathfrak{p}]}$. Hence $A_{[\mathfrak{p}]} = AR_{[\mathfrak{p}]}$. □

3. Star operations induced by saturations of submodules

In this section, we use the notion of saturation along with the notion of star operation to investigate properties of ring extensions. First we recall the definition of a star operation as given by Knebusch and Kaiser in [6, Definition 1, p. 139]. Let $R \subseteq S$ be a ring extension. A map $\star$: $\mathcal{J}(R, S) \to \mathcal{J}(R, S)$, where $\mathcal{J}(R, S)$ is the set of all R-submodules of S, is called *star operation on* $R \subseteq S$ if the following conditions are satisfied for all A, $B \in \mathcal{J}(R, S)$.

(c_1) $A \subseteq A^\star$.
(c_2) If $A \subseteq B$, then $A^\star \subseteq B^\star$.
(c_3) $(A^\star)^\star = A^\star$.
(c_4) $AB^\star \subseteq (AB)^\star$.

A star operation $\star$ on a ring extension $R \subseteq S$ is said to be *strict* if $R^\star = R$. For more on star operation of ring extension, see [6, pages 139 - 164].

In the next two results, we give examples of star operations induced by saturations.

Proposition 3.1. *Let $R \subseteq S$ be a ring extension, and let $\star : \mathcal{J}(R,S) \to \mathcal{J}(R,S)$ be the map defined by $A^\star = \bigcap_{\mathfrak{p}\in\mathcal{P}} A_{[\mathfrak{p}]}$, where $\mathcal{P}$ is a set of prime ideals of R. Then*

1. *The map $\star$ is a star operation on the extension $R \subseteq S$.*
2. *$A_{[\mathfrak{p}]} = (A^\star)_{[\mathfrak{p}]}$ for each $A \in \mathcal{J}(R,S)$ and each $\mathfrak{p} \in \mathcal{P}$.*
3. *If $R = \bigcap_{\mathfrak{p}\in\mathcal{P}} R_{[\mathfrak{p}]}$, then $\mathfrak{p}^\star = \mathfrak{p}$ for each $\mathfrak{p} \in \mathcal{P}$.*

Proof. (1) Let A, B and J be elements of $\mathcal{J}(R,S)$. From the definition of $\star$, it is clear that $A \subseteq A^\star$, and $A^\star \subseteq B^\star$ if $A \subseteq B$. Furthermore, $JA^\star = J(\bigcap_{\mathfrak{p}\in\mathcal{P}} A_{[\mathfrak{p}]}) \subseteq JA_{[\mathfrak{p}]}$ for each $\mathfrak{p} \in \mathcal{P}$. It follows that $JA^\star \subseteq \bigcap_{\mathfrak{p}\in\mathcal{P}} JA_{[\mathfrak{p}]} = (JA)^\star$.

Now we show that $(A^\star)^\star = A^\star$. Let $x \in (A^\star)^\star$. Then for each $\mathfrak{p}_0 \in \mathcal{P}$, we have $x \in (A^\star)_{[\mathfrak{p}_0]}$. So there exists $t_0 \in R\backslash\mathfrak{p}_0$ such that $t_0 x \in A^\star = \bigcap_{\mathfrak{p}\mathcal{P}} A_{[\mathfrak{p}]}$. In particular $t_0 x \in A_{[\mathfrak{p}_0]}$. Therefore, $t_0 s_0 x \in A$ for some $s_0 \in R \setminus \mathfrak{p}_0$. It follows that $x \in A_{[p_0]}$, since $t_0 s_0 \in R \setminus \mathfrak{p}_0$. Since $\mathfrak{p}_0$ was chosen arbitrarily in $\mathcal{P}$, we have $x \in \bigcap_{\mathfrak{p}} A_{[\mathfrak{p}]} = A^\star$. Thus, $(A^\star)^\star \subseteq A^\star$. The containment $A^\star \subseteq (A^\star)^\star$ is clear. Hence $(A^\star)^\star = A^\star$.

(2) Let $A \in \mathcal{J}(R,S)$, and let $\mathfrak{p} \in \mathcal{P}$. By part (1), we have $A \subseteq A^\star$. So $A_{[\mathfrak{p}]} \subseteq (A^\star)_{[\mathfrak{p}]}$. It remains to show that $(A^\star)_{[\mathfrak{p}]} \subseteq A_{[\mathfrak{p}]}$. Notice that for each $\mathfrak{p} \in \mathcal{P}$, we have $A^\star \subseteq \bigcap_{\mathfrak{q}\in\mathcal{P}} A_{[\mathfrak{q}]} \subseteq A_{[\mathfrak{p}]}$. Thus $(A^\star)_{[\mathfrak{p}]} \subseteq (A_{[\mathfrak{p}]})_{[\mathfrak{p}]} = A_{[\mathfrak{p}]}$.

(3) Let $\mathfrak{p}_0 \in \mathcal{P}$. Then by part (1), we have $\mathfrak{p}_0 \subseteq \mathfrak{p}_0^\star$. Now we show the containment $\mathfrak{p}_0^\star \subseteq \mathfrak{p}_0$. Let $x \in \mathfrak{p}_0^\star$. Then $x \in \bigcap_{\mathfrak{p}\in\mathcal{P}} p_{0[\mathfrak{p}]} \subseteq \bigcap_{\mathfrak{p}\in\mathcal{P}} R_{[\mathfrak{p}]} = R$. Furthermore, there exists $t_0 \in R \setminus \mathfrak{p}_0$ such that $t_0 x \in \mathfrak{p}_0$. It follows that $x \in \mathfrak{p}_0$, since $\mathfrak{p}_0$ is prime and $t_0 \notin \mathfrak{p}$. This shows that $\mathfrak{p}_0^\star \subseteq \mathfrak{p}_0$. Hence $\mathfrak{p} = \mathfrak{p}^\star$. □

Proposition 3.2. *Let $R \subseteq S$ be a ring extension, and let $\star : \mathcal{J}(R,S) \to \mathcal{J}(R,S)$ be the map defined by $A^\star = \bigcap_{\mathfrak{p}\in\mathcal{P}} AR_{[\mathfrak{p}]}$, where $\mathcal{P}$ is a set of prime ideals of R. Then*

1. *The map $\star$ is a star operation on the extension $R \subseteq S$.*
2. *$A_{[\mathfrak{p}]} = (A^\star)_{[\mathfrak{p}]}$ for each $A \in \mathcal{J}(R,S)$ and each $\mathfrak{p} \in \mathcal{P}$.*
3. *If $R = \bigcap_{\mathfrak{p}\in\mathcal{P}} R_{[\mathfrak{p}]}$, then $\mathfrak{p}^\star = \mathfrak{p}$ for each $\mathfrak{p} \in \mathcal{P}$.*

Proof. (1) Let A, B and J be elements of $\mathcal{J}(R,S)$. From the definition of $\star$, it is clear that $A \subseteq A^\star$. Also, if $A \subseteq B$, then $AR_{[\mathfrak{p}]} \subseteq BR_{[\mathfrak{p}]}$ for each prime ideal $\mathfrak{p}$ of $\mathcal{P}$. Hence $A^\star = \bigcap_{\mathfrak{p}\in\mathcal{P}} AR_{[\mathfrak{p}]} \subseteq \bigcap_{\mathfrak{p}\in\mathcal{P}} BR_{[\mathfrak{p}]} = B^\star$. Furthermore, $JA^\star = J(\bigcap_{\mathfrak{p}\in\mathcal{P}} AR_{[\mathfrak{p}]}) \subseteq JAR_{[\mathfrak{p}]}$ for each $\mathfrak{p} \in \mathcal{P}$. It follows that $JA^\star \subseteq \bigcap_{\mathfrak{p}\in\mathcal{P}} JAR_{[\mathfrak{p}]} = (JA)^\star$. Notice that if $\mathfrak{q} \in \mathcal{P}$, then $(A^\star)^\star = \bigcap_{\mathfrak{p}\in\mathcal{P}} A^\star R_{[\mathfrak{p}]} \subseteq A^\star R_{[\mathfrak{q}]} = (\bigcap_{\mathfrak{p}\in\mathcal{P}} AR_{[\mathfrak{p}]})R_{[\mathfrak{q}]} \subseteq (AR_{[\mathfrak{q}]})R_{[\mathfrak{q}]} \subseteq AR_{[\mathfrak{q}]}$. It follows that $(A^\star)^\star \subseteq AR_{[\mathfrak{q}]}$ for each $\mathfrak{q} \in \mathcal{P}$. Thus $(A^\star)^\star = \bigcap_{\mathfrak{q}\in\mathcal{P}} AR_{\mathfrak{q}} = A^\star$.

(2) Let $A \in \mathcal{J}(R,S)$, and let $\mathfrak{p} \in \mathcal{P}$. By part (1), we have $A \subseteq A^\star$. It follows that $A_{[\mathfrak{p}]} \subseteq (A^\star)_{[\mathfrak{p}]}$. It remains to show that $(A^\star)_{[\mathfrak{p}]} \subseteq A_{[\mathfrak{p}]}$. We have $(A^\star)_{[\mathfrak{p}]} = (\bigcap_{\mathfrak{q}\in\mathcal{P}} AR_{[\mathfrak{q}]})_{[\mathfrak{p}]} \subseteq (AR_{[\mathfrak{p}]})_{[\mathfrak{p}]}$. But by Proposition 2.1(1), we have $AR_{[\mathfrak{p}]} \subseteq A_{[\mathfrak{p}]}$. Thus $(A^\star)_{[\mathfrak{p}]} \subseteq (A_{[\mathfrak{p}]})_{[\mathfrak{p}]} = A_{[\mathfrak{p}]}$. This shows that $(A^\star)_{[\mathfrak{p}]} = A_{[\mathfrak{p}]}$.

(3) Let $\mathfrak{p}_0 \in \mathcal{P}$. Then by part (1), we have $\mathfrak{p}_0 \subseteq \mathfrak{p}_0^\star$. Now we show the containment $\mathfrak{p}_0^\star \subseteq \mathfrak{p}_0$. We have $\mathfrak{p}_0^\star = \bigcap_{\mathfrak{p}\in\mathcal{P}} \mathfrak{p}_0 R_{[\mathfrak{p}]}$. But by Proposition 2.1(1), we have $\mathfrak{p}_0 R_{[\mathfrak{p}]} \subseteq \mathfrak{p}_{0[\mathfrak{p}]}$ for each $\mathfrak{p} \in \mathcal{P}$. Thus $\mathfrak{p}_0^\star \subseteq \bigcap_{\mathfrak{p}\in\mathcal{P}} \mathfrak{p}_{0[\mathfrak{p}]}$. Furthermore, by Proposition 3.1(3), we have $\bigcap_{\mathfrak{p}\in\mathcal{P}} \mathfrak{p}_{0[\mathfrak{p}]} = \mathfrak{p}_0$. Therefore, $\mathfrak{p}_{[0]}^\star = \mathfrak{p}_{[0]}$. □

Let $R \subseteq S$ be a ring extension, and let X be an indeterminate over S. For each element f of $S[X]$, and each subring L of S containing R, we denote by $c_L(f)$ the L-submodule of S generated by the coefficients of f. Let $T = \{g \in S[X] : \ c_S(g) = S\}$. Then T is a multiplicative subset of $S[X]$ [5,]. We denote by $S(X)$ the quotient ring $(S[X])_T$. Hence $S(X)$ is the set of all elements of the form $\frac{f}{g}$ with $f, g \in S[X]$ such that $c_S(g) = S$ (for properties of the ring $S(X)$, see [5, p. 410]). A subring B of $S(X)$ is said to be *Kronecker subring* of $S(X)$ if $X \in B$, and for every $f = a_0 + a_1X + \cdots a_nX^n \in S[X]$ with $c_S(f) = S$ the elements $\frac{a_j}{f} \in S(X)$ with $j = o, \dots n$, are contained in B [6, Definition 4, p. 127].

For each star operation $\star$ on a ring extension $R \subseteq S$, we denote by $\mathrm{Kr}(\star)$ the set of all elements $\frac{f}{g} \in S(X)$ with $(c_R(f)H)^\star \subseteq (c_R(g)H)^\star$ for some finitely generated S-regular R-submodule H of S. The set $\mathrm{Kr}(\star)$ is a Kronecker subring of $S(X)$[6, Theorem 3.10, p. 143], and the extension $\mathrm{Kr}(\star) \subseteq S(X)$ is Bézout [6, Theorem 1.5, p. 128].

Remark 3.3. Let $R \subseteq S$ be a ring extension, and let $\mathcal{P}$ be a set of prime ideals of R. Let $\star_1, \star_2 : \mathcal{J}(R,S) \to \mathcal{J}(R,S)$ be two maps defined for each $A \in \mathcal{J}(R,S)$ by $A^{\star_1} = \bigcap_{[\mathfrak{p}]\in\mathcal{P}} A_{[\mathfrak{p}]}$ and $A^{\star_2} = \bigcap_{[\mathfrak{p}]\in\mathcal{P}} AR_{[\mathfrak{p}]}$ respectively. Then by Proposition 3.1, $\star_1$ is a star operation on $R \subseteq S$. Also, by Proposition 3.2, the map $\star_2$ is also a star operation on $R \subseteq S$. Let $\frac{f}{g} \in \mathrm{Kr}(\star_2)$. Then there exists a finitely generated S-regular R-submodule H of S such that $c_R(f)H \subseteq (c_R(f)H)^{\star_2} \subseteq (c_R(g)H)^{\star_2}$. But by Proposition 2.1(1), $(c_R(g)H)^{\star_2} \subseteq (c_R(g)H)^{\star_1}$. It follows that $(c_R(f)H) \subseteq (c_R(g)H)^{\star_1}$, and hence $(c_R(f)H)^{\star_1} \subseteq (c_R(g)H)^{\star_1}$. Therefore, $\frac{f}{g} \in \mathrm{Kr}(\star_1)$. This shows that $\mathrm{Kr}(\star_2) \subseteq \mathrm{Kr}(\star_1)$.

Remark 3.4. Let $R \subseteq S$ be a ring extension, and let $\mathcal{M}$ be the set of all maximal ideals of R. Then by [7, Remark 5.5, p. 50], we have $A^{b_1} = \bigcap_{\mathfrak{p}\in\mathcal{M}} A_{[\mathfrak{p}]} = \bigcap_{\mathfrak{p}\in\mathcal{M}} AR_{[\mathfrak{p}]} = A^{b_2}$. Hence $\mathrm{Kr}(b_1) = \mathrm{Kr}(b_2)$.

Proposition 3.5. *Let $R \subseteq S$ be a ring extension, and let $\mathcal{P}$ be a set of prime ideals of R. Let $\star_1$ and $\star_2$ be the two star operations on $R \subseteq S$ defined for each R-submodule A of S by $A^{\star_1} = \bigcap_{\mathfrak{p} \in \mathcal{M}} A_{[\mathfrak{p}]}$ and $A^{\star_2} = \bigcap_{\mathfrak{p} \in \mathcal{M}} AR_{[\mathfrak{p}]}$. If the extension $R \subseteq S$ is Prüfer, then $\mathrm{Kr}(\star_1) = \mathrm{Kr}(\star_2)$.*

Proof. Suppose that the extension $R \subseteq S$ is Prüfer. Let $\frac{f}{g} \in \mathrm{Kr}(\star_1)$. Then there exists a finitely generated S-regular R-submodule H of S such that $c_R(f)H \subseteq (c_R(f)H)^{\star_1} \subseteq (c_R(g)H)^{\star_1}$. Furthermore, $(c_R(g)H)S = S$, since $c_R(g)S = S$ and $HS = S$. Thus $c_R(g)H$ is an S-regular finitely generated R-submodule of S. Therefore, by Proposition 2.11, we have $(c_R(g)H)^{\star_2} = (c_R(g)H)^{\star_1}$. It follows from the relation $c_R(f)H \subseteq (c_R(g)H)^{\star_1}$ that $c_R(f)H \subseteq (c_R(g)H)^{\star_2}$. Therefore,

$$(c_R(f)H)^{\star_2} \subseteq ((c_R(g)H)^{\star_2})^{\star_2} = (c_R(g)H)^{\star_2}.$$

Thus $\frac{f}{g} \in \mathrm{Kr}(\star_2)$. This shows that $\mathrm{Kr}(\star_1) \subseteq \mathrm{Kr}(\star_2)$. But by Remark 3.3, we have $\mathrm{Kr}(\star_2) \subseteq \mathrm{Kr}(\star_1)$. Hence $\mathrm{Kr}(\star_1) = \mathrm{Kr}(\star_2)$. □

Remark 3.6. Let $R \subseteq S$ be a ring extension, and let A, B be two R-submodules of S. By definition of $(A : B)$, we have $(A : B)B \subseteq A$. Furthermore, the containment $(A : B) \subseteq R$ implies that $(A : B)B \subseteq B$. Thus $(A : B)B \subseteq A \cap B$.

Lemma 3.7. *Let $R \subseteq S$ be a ring extension, and let A be an R-submodule of S. For each S-invertible R-submodule I of S, we have $A \cap I = (A : I)I$.*

Proof. Let A be an R-submodule of S, and let I be an S-invertible R-submodule of S. Let $\mathfrak{p}$ be a prime ideal of R. By [7, Proposition 2.3, p. 97], there exists $w_\mathfrak{p} \in I$ such that the $R_\mathfrak{p}$-module $I_\mathfrak{p}$ is generated by $\frac{w_\mathfrak{p}}{1}$. Let $x \in (A \cap I)_\mathfrak{p} = A_\mathfrak{p} \cap I_\mathfrak{p}$. Then $x = \frac{rw_\mathfrak{p}}{t}$ with $r \in R$ and $t \in R \setminus \mathfrak{p}$. Since $\frac{rw_\mathfrak{p}}{t} \in A_\mathfrak{p}$, we have $\frac{r}{t} \in (A_\mathfrak{p} :_{R_\mathfrak{p}} I_\mathfrak{p})$. Furthermore, I is a finitely generated R-submodule of S [7, Remark 1.10 (a)]. Hence $(A_\mathfrak{p} :_{R_\mathfrak{p}} I_\mathfrak{p}) = (A :_R I)_\mathfrak{p}$ [1, Corollary 3.15 p.43]. Thus $\frac{r}{t} \in (A :_R I)_\mathfrak{p}$. It follows that $x = \frac{rw_\mathfrak{p}}{t} \in (A :_R I)_\mathfrak{p} I_\mathfrak{p} = ((A :_R I)I)_\mathfrak{p}$. This shows that $(A \cap I)_\mathfrak{p} \subseteq ((A :_R I)I)_\mathfrak{p}$. On the other hand, the containment $(A :_R I)I \subseteq A \cap I$ follows from Remark 3.6. Hence $((A :_R I)I)_\mathfrak{p} \subseteq (A \cap I)_\mathfrak{p}$. Therefore, $((A :_R I)I)_\mathfrak{p} = (A \cap I)_\mathfrak{p}$. It follows from [7, Lemma 1.1, p. 85] that $A \cap I = (A : I)I$, since $\mathfrak{p}$ was chosen arbitrarily in R. □

Remark 3.8. Let $R \subseteq S$ be a tight ring extension, and let A be an S-regular R-submodule of A. Then for each element x of A, there exists an S-invertible R-submodule E of S such that such that $x \in E \subseteq A$.

Proof. Let A be an S-regular R-submodule of S. Then $AS = S$. Therefore, there exist $a_1, \dots, a_n \in A$ and $s_1, \dots, s_n \in S$ such that $a_1 s_1 + \cdots + a_n s_n = 1$. Let $x \in A$. Since the extension $R \subseteq S$ is tight, there exists $u \in S$ such that $xu \in R$. Let E be the R-submodule of S generated by $x, a_1, \dots, a_n$. Then E is S-invertible, and E^{-1} is the R-submodule of S generated by $u, s_1, \dots, s_n$. Furthermore, $x \in E \subseteq A$. □

Remark 3.9. Let $R \subseteq S$ be a tight ring extension, and let B be a subring of S containing R. Let $x \in B$. If $x \in R$, then $xR \subseteq R$. Thus $R \subseteq (R : x)$, and hence $(R : x)S = S$. If $x \in S \setminus R$, then by the definition of a tight extension, there exists a S-invertible ideal I of R such that $xI \subseteq R$. Hence $I \subseteq (R : x)$. Thus $IS \subseteq (R : x)S$. But by [7, Remarks 1.10(d), p. 90], we have $IS = S$. Hence $(R : x)S = S$. It follows from [7, Theorem 3.13, p. 37] and [7, Proposition 1.15, p. 91] that if A and B are two S-regular R-submodules of S, then $A \cap B$ and $(A : B)$ are also S-regular R-submodules of S.

The following theorem is inspired by [2, Theorem 4], which is a result due to D.D. Anderson. We consider a tight ring extension $R \subseteq S$ and we characterize star operations $\star$ on $R \subseteq S$ satisfying the condition $(A \cap B)^\star = A^\star \cap B^\star$ where A and B are S-regular R-submodule of S.

Theorem 3.10. *Let $R \subseteq S$ be a tight ring extension, and let $\mathcal{P}$ be a set of prime ideals of R such that each S-regular ideal of R is contained in some $\mathfrak{p} \in \mathcal{P}$. Then for each star operation $\star$ on the ring extension $R \subseteq S$ satisfying $R = R^\star$, the following statements are equivalent.*

(1) $A^\star = \bigcap_{\mathfrak{p} \in \mathcal{P}} A_{[\mathfrak{p}]}$ *for each S-regular R-submodules A of R.*

(2) (a) $R = \bigcap_{\mathfrak{p} \in \mathcal{P}} R_{[\mathfrak{p}]}$.

 (b) $(A \cap B)^\star = A^\star \cap B^\star$ *for all S-regular R-submodules A, B of R.*

(3) (a) $R = \bigcap_{\mathfrak{p} \in \mathcal{P}} R_{[\mathfrak{p}]}$.

 (b) $(A :_R B)^\star = (A^\star :_R B^\star)$ *for all S-regular R-submodule A, B of R with B finitely generated.*

Proof. (1) ⇒ (2) Since R is an S-regular R-submodule of R, we have $R^\star = \bigcap_{\mathfrak{p} \in \mathcal{P}} R_{[\mathfrak{p}]}$. It follows that $R = \bigcap_{\mathfrak{p} \in \mathcal{P}} R_{[\mathfrak{p}]}$, since by hypothesis, we have $R = R^\star$. Let A and B be two S-regular R-submodules of S. Then by Remark 3.9, $A \cap B$ is also an S-regular R-submodule of S. Thus by hypothesis, we have $(A \cap B)^\star = \bigcap_{\mathfrak{p} \in \mathcal{P}} (A \cap B)_{[\mathfrak{p}]}$. But by Proposition 2.1(2), $(A \cap B)_{[\mathfrak{p}]} = A_{[\mathfrak{p}]} \cap B_{[\mathfrak{p}]}$. Therefore,

$$(A \cap B)^\star = \bigcap_{\mathfrak{p} \in \mathcal{P}} (A \cap B)_{[\mathfrak{p}]} = \bigcap_{\mathfrak{p} \in \mathcal{P}} (A_{[\mathfrak{p}]} \cap B_{[\mathfrak{p}]}) = \bigcap_{\mathfrak{p} \in \mathcal{P}} A_{[\mathfrak{p}]} \cap \bigcap_{\mathfrak{p} \in \mathcal{P}} B_{[\mathfrak{p}]} = A^\star \cap B^\star.$$

(2) $\Rightarrow$ (1) Let A be an S-regular R-submodule of S. Then $AS = S$. It follows that $A^\star S = S$, since $A \subseteq A^\star$. Let $x \in A^\star$. Then by Remark 3.8, there exists an S-invertible R-submodule I of S such that $x \subseteq I \subseteq A^\star$. Since $I \subseteq A^\star$, we have $I^\star = A^\star \cap I^\star$. But by hypothesis, we have $A^\star \cap I^\star = (A \cap I)^\star$. Also, by Lemma 3.7, we have $A \cap I = (A : I)I$. It follows that $I^\star = ((A : I)I)^\star$. But by [6, Proposition 4.1(b), p. 146], we have $((A : I)I)^\star = (A : I)^\star I$. This shows that $I^\star = (A : I)^\star I$, and so $I \subseteq (A : I)^\star I$. By multiplying the last relation by I^{-1}, we get $R \subseteq (A : I)^\star$. Furthermore, by [7, Remarks 1.10(d)], I is an S-regular R-submodule of S. It follows from Remark 3.9 that $(A : I)$ is an S-regular ideal of R. If $(A : I)$ is a proper ideal of R, then by hypothesis, $(A : I) \subseteq \mathfrak{p}_0$ for some $\mathfrak{p}_0 \in \mathcal{P}$ of R. In this case $S = (A : I)S \subseteq \mathfrak{p}_0 S$. Hence $\mathfrak{p}_0 S = S$. So $\mathfrak{p}_0$ is an S-regular ideal of R. It follows that $\mathfrak{p}_0^\star = \bigcap_{\mathfrak{p}\in\mathcal{P}} p_{0[\mathfrak{p}]}$. Therefore, as in the the proof of Proposition 3.1(3), we have $\mathfrak{p}_0 = \mathfrak{p}_0^\star$. Hence $R \subseteq (A : I)^\star \subseteq \mathfrak{p}_0^\star = \mathfrak{p}_0$; which is impossible. Thus $R = (A : I)$. Let $\mathfrak{p} \in \mathcal{P}$ and let $t \in R \setminus \mathfrak{p}$. Then $t \in (A : I) = R$. It follows that $tI \subseteq A$. In particular, $tx \in A$. Thus $x \in A_{[\mathfrak{p}]}$. This shows that $A^\star \subseteq \bigcap_{[\mathfrak{p}\in\mathcal{P}]} A_{[\mathfrak{p}]}$.

For the containment $\bigcap_{[\mathfrak{p}\in\mathcal{P}]} A_{[\mathfrak{p}]} \subseteq A^\star$, let $A' = \bigcap_{\mathfrak{p}\in\mathcal{P}} A_{[\mathfrak{p}]}$. Then $A \subseteq A'$. It follows that $A'S = S$. Let $y \in A'$. Then by Remark 3.8, there exists an S-invertible R-submodule J of S such that $y \in J \subseteq A'$. Let $\mathfrak{p} \in \mathcal{P}$. Then by Proposition 3.1(2), we have $A'_{[\mathfrak{p}]} = A_{[\mathfrak{p}]}$. Thus $J \subseteq A' \subseteq A'_{[\mathfrak{p}]} = A_{[\mathfrak{p}]}$. But J is a finitely generated R-submodule of S [7, Remarks 1.10(a), p. 90]. So there exist $u_1, \ldots, u_\ell \in S$ such that $J = (u_1, \ldots, u_\ell)R$. For $1 \leqslant i \leqslant \ell$, let $t_i \in R \setminus \mathfrak{p}$ such that $t_i u_i \in A$, and let $t = \prod_{i=1}^{\ell} t_i$. Then $t \in R\setminus\mathfrak{p}$ and $tJ \subseteq A$. It follows that $(A : J) \cap (R\setminus\mathfrak{p}) \neq \varnothing$. Notice that J is an S-regular R-submodule of S, since each S-invertible R-submodule of S is S-regular[7, Remarks 1.10(d), p. 90]. Therefore, by Remark 3.9, we conclude that $(A : J)$ is an S-regular ideal of R. It follows from the hypothesis and the relation $(A : J) \cap (R \setminus \mathfrak{p}) \neq \varnothing$ for each $\mathfrak{p} \in \mathcal{P}$, that $(A : J)$ is not a proper ideal of R. Hence $(A : J) = R$, and thus $(A : J)^\star = R^\star$. Furthermore, $J^\star = (JR)^\star = JR^\star$ [6, Proposition 4.1, p. 146]. Therefore, $J^\star = JR^\star = J(A : J)^\star = (J(A : J))^\star = (J \cap A)^\star$. It follows that $y \in J \subseteq J^\star \subseteq A^\star$. This shows that $A' = \bigcap_{\mathfrak{p}\in\mathcal{P}} A_{[\mathfrak{p}]} \subseteq A^\star$. Therefore, $A^\star = \bigcap_{\mathfrak{p}\in\mathcal{P}} A_{[\mathfrak{p}]}$.

(1) $\Rightarrow$ (3) We have $R^\star = \bigcap_{\mathfrak{p}\in\mathcal{P}} R_{[\mathfrak{p}]}$, since R is an S-regular R-submodule of S. It follows that $R = \bigcap_{\mathfrak{p}\in\mathcal{P}} R_{[\mathfrak{p}]}$, since by hypothesis, we have $R = R^\star$. Let A and B be two S-regular R-submodules of S, and suppose that B is finitely generated. Then by Remark 3.9, the ideal $(A : B)$ of R is also S-regular. Thus by the hypothesis, we have

$(A : B)^\star = \bigcap_{\mathfrak{p}\in\mathcal{P}}(A : B)_{[\mathfrak{p}]}$. It follows from Proposition 2.1(4) that $(A : B)^\star = \bigcap_{\mathfrak{p}\in\mathcal{P}}(A_{[\mathfrak{p}]} : B_{[\mathfrak{p}]}) \subseteq (\bigcap_{\mathfrak{p}\in\mathcal{P}} A_{[\mathfrak{p}]} : \bigcap_{\mathfrak{p}\in\mathcal{P}} B_{[\mathfrak{p}]}) = (A^\star : B^\star)$. Now we show that $(A^\star : B^\star) \subseteq (A : B)^\star$. First notice that the containment $(A^\star : B^\star) \subseteq (A^\star : B)$ is always true. Let $d \in (A^\star : B^\star)$. Then for each $\mathfrak{p} \in \mathcal{P}$, we have $d \in (A^\star : B) \subseteq (A^\star : B)_{[\mathfrak{p}]}$. But by Proposition 2.1(4), we have $(A^\star : B)_{[\mathfrak{p}]} = ((A^\star)_{[\mathfrak{p}]} : B_{[\mathfrak{p}]})$. Thus $d \in ((A^\star)_{[\mathfrak{p}]} : B_{[\mathfrak{p}]})$. Furthermore, $A_{[\mathfrak{p}]} \subseteq (A^\star)_{[\mathfrak{p}]} = (\bigcap_{\mathfrak{q}\in\mathcal{P}} A_{[\mathfrak{q}]})_{[\mathfrak{p}]} \subseteq (A_{[\mathfrak{p}]})_{[\mathfrak{p}]} = A_{[\mathfrak{p}]}$. Thus $(A^\star)_{[\mathfrak{p}]} = A_{[\mathfrak{p}]}$. Hence $d \in (A_{[\mathfrak{p}]} : B_{[\mathfrak{p}]})$. It follows from Proposition 2.1(4) that $d \in (A : B)_{[\mathfrak{p}]}$. This shows that $d \in \bigcap_{\mathfrak{p}\in\mathcal{P}}(A : B)_{[\mathfrak{p}]} = (A : B)^\star$. Hence $(A^\star : B^\star) \subseteq (A : B)^\star$, since d was chosen arbitrarily in $(A^\star : B^\star)$. Thus $(A^\star : B^\star) = (A : B)^\star$.

(3) $\Rightarrow$ (1) Let A be an S-regular R-submodule of S. Let $x \in A^\star$. The R-submodule $A^\star$ is S-regular, since $A \subseteq A^\star$. Therefore, by Remark 3.8, there exits an S-invertible R-submodule I of S such that $x \in I \subseteq A^\star$. The containment $I \subseteq A^\star$ implies $I^\star \subseteq A^\star$. Thus $R = (A^\star : I^\star)$. Furthermore, I is a finitely generated R-submodule of S [7, Remark 1.10 (a)]. Thus $R = (A^\star : I^\star) = (A : I)^\star$. Suppose that $(A : I)$ is a proper ideal of R. Then by hypothesis, $(A : I) \subseteq \mathfrak{p}_0$ for some $\mathfrak{p}_0 \in \mathcal{P}$ of R. Thus $S = (A : I)S \subseteq \mathfrak{p}_0 S$. Hence $\mathfrak{p}_0 S = S$. So $\mathfrak{p}_0$ is an S-regular ideal of R. It follows that $\mathfrak{p}_0^\star = \bigcap_{\mathfrak{p}\in\mathcal{P}} p_{0[\mathfrak{p}]}$. Thus, as in the the proof of Proposition 3.1(3), we have $\mathfrak{p}_0 = \mathfrak{p}_0^\star$. Hence $R \subseteq (A : I)^\star \subseteq \mathfrak{p}_0^\star = \mathfrak{p}_0$; which is impossible. Thus $R = (A : I)$. The rest of the proof is identical to the proof of (2) $\Rightarrow$ (1). $\square$

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