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Green's relations on the seminearring of full hypersubstitutions of type (n)

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Abstract. Hypersubstitutions are mappings which are used to define hyperidentities and solid varieties. In this paper we will show that the set of all hypersubstitutions of a given type forms a seminearring. We will give a full characterization of Green's relation $\mathcal{R}$ on a sub-seminearring of the seminearring $Hyp(n)$ of all hypersubstitutions of type (n).

1. Introduction

Hypersubstitutions were introduced to make precise the concept of a hyperidentity and generalizations to M-hyperidentities. Let $\tau = (n_i)_{i \in I}$ be a type indexed by a set I, with operation symbols f_i of arity $n_i \in \mathbb{N}$. Let $X = \{x_1, x_2, \ldots\}$ be a countably infinite set of variables and let $X_n = \{x_1, \ldots, x_n\}$ be a finite set. We denote by $W_\tau(X_n)$ the set of all n-ary terms of type τ over the alphabet X_n and by $W_\tau(X)$ the set of all terms of type τ.
An identity $s \approx t, s, t \in W_\tau(X)$, of type τ is called a *hyperidentity* of a variety V of type τ if for every substitution of n-ary terms for the n-ary operation symbols in $s \approx t$, the resulting identity holds in V. This shows that we are interested in mappings which associate to every operation symbol f_i of a given type τ a term $\sigma(f_i)$ of type τ of the same arity as f_i. Any such map is called a *hypersubstitution of type* τ.

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Hypersubstitutions can be uniquely extended to mappings $\hat{\sigma} : W_\tau(X) \to W_\tau(X)$ which are inductively defined by the following steps:

(i) $\hat{\sigma}[x] := x$ if $x \in X$,

(ii) $\hat{\sigma}[t] := \sigma(f_i)(\hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_{n_i}])$ if $t = f_i(t_1, \ldots, t_{n_i})$.

Using this extension we can define a binary operation $\circ_h$ on the set $Hyp(\tau)$ of all hypersubstitutions of type τ by $\sigma_1 \circ_h \sigma_2 := \hat{\sigma}_1 \circ \sigma_2$, where $\circ$ is the usual composition of operations. By σ_{id} we denote the identity hypersubstitution, mapping each operation symbol f_i to the term $f_i(x_1, \ldots x_{n_i})$. This gives the monoid $(Hyp(\tau); \circ_h, \sigma_{id})$.

If M is any submonoid of the monoid $Hyp(\tau)$, then an identity $u \approx v$ of a variety V is called an *M-hyperidentity* of V if for every $\sigma \in M$ the equation $\hat{\sigma}[u] \approx \hat{\sigma}[v]$ holds in V. A variety V is called *M-solid* if every identity of V is an M-hyperidentity in V. The collection of all M-solid varieties of type τ forms a complete sublattice of the lattice of all varieties of type τ. Actually, there is a Galois connection between submonoids of $Hyp(\tau)$ and complete sublattices of the lattice of all varieties of type τ. For more background see [3]. This shows the importance of studying the properties of the monoid $Hyp(\tau)$ and its submonoids. In [2] the authors started to investigate the semigroup properties of the monoid $Hyp(2)$ of all hypersubstitutions of type $\tau = (2)$, especially Green's relations on $Hyp(2)$. We want to continue these investigations with $Hyp(n)$, the monoid of all hypersubstitutions of type $\tau = (n)$, and a particular submonoid of $Hyp(n)$, the monoid of all so-called *full hypersubstitutions.*

It turns out that one can define a second binary operation on $Hyp(\tau)$ such that $Hyp(\tau)$ forms a seminearring. By

$$(\sigma_1 + \sigma_2)(f_i) := \sigma_2(f_i)(\sigma_1(f_i), \ldots, \sigma_1(f_i))$$

we define a hypersubstitution which maps the n_i-ary operation symbol f_i to the n_i-ary term $\sigma_2(f_i)(\sigma_1(f_i), \ldots, \sigma_1(f_i))$ for every $i \in I$. The term $\sigma_2(f_i)(\sigma_1(f_i), \ldots, \sigma_1(f_i))$ is n_i-ary. Therefore $\sigma_1 + \sigma_2$ is a hypersubstitution of $Hyp(\tau)$. We show that the operation $+$ is associative. Indeed, we have

$$\begin{gathered}((\sigma_1 + \sigma_2) + \sigma_3)(f_i) = \sigma_3(f_i)((\sigma_1 + \sigma_2)(f_i), \ldots, (\sigma_1 + \sigma_2)(f_i)) = \\ = \sigma_3(f_i)(\sigma_2(f_i)(\sigma_1(f_i), \ldots, \sigma_1(f_i)), \ldots, \sigma_2(f_i)(\sigma_1(f_i), \ldots, \sigma_1(f_i))) = \\ = \sigma_3(f_i)(\sigma_2(f_i), \ldots, \sigma_2(f_i))(\sigma_1(f_i), \ldots, \sigma_1(f_i)) = \\ = (\sigma_2 + \sigma_3)(f_i)(\sigma_1(f_i), \ldots, \sigma_1(f_i)) = (\sigma_1 + (\sigma_2 + \sigma_3))(f_i).\end{gathered}$$

Here we used properties of the superposition of terms. In a similar way we prove that $\sigma \circ_h (\sigma_1 + \sigma_2) = (\sigma \circ_h \sigma_1) + (\sigma \circ_h \sigma_2)$.

Indeed, we have

$$\begin{gathered}(\sigma \circ_h (\sigma_1 + \sigma_2))(f_i) = \hat{\sigma}[(\sigma_1 + \sigma_2)(f_i)] = \\ = \hat{\sigma}[\sigma_2(f_i)(\sigma_1(f_i), \ldots, \sigma_1(f_i))] = \\ = \hat{\sigma}[\sigma_2(f_i)](\hat{\sigma}[\sigma_1(f_i)], \ldots, \hat{\sigma}[\sigma_1(f_i)]) = \\ = (\sigma \circ_h \sigma_2)(f_i)((\sigma \circ_h \sigma_1)(f_i), \ldots, (\sigma \circ_h \sigma_1)(f_i)) \\ ((\sigma \circ_h \sigma_1) + (\sigma \circ_h \sigma_2))(f_i)\end{gathered}$$

if we use that hypersubstitution and superposition are permutable. This shows the left distributivity.
The following counterexample shows that the right distributive identity is not satisfied.
Assume that $\tau = (2)$, with a binary operation symbol f, and that $\sigma_1, \sigma_2, \sigma_3$ are defined by $\sigma_1(f) = f(x, y)$, $\sigma_2(f) = f(y, x)$, $\sigma_3(f) = f(x, f(y, y))$. Then

$$\begin{gathered}(\sigma_1 + \sigma_2)(f) = f(f(x, y), f(x, y)), \\ ((\sigma_1 + \sigma_2) \circ_h \sigma_3)(f) = (\sigma_1 + \sigma_2)\hat{}\,[f(x, f(y, y))], \\ = f(f(x, f(f(y, y), f(y, y))), f(x, f(f(y, y), f(y, y)))), \\ (\sigma_1 \circ_h \sigma_3)(f) = f(x, f(y, y)), \\ (\sigma_2 \circ_h \sigma_3)(f) = f(f(y, y), x), \\ ((\sigma_1 \circ_h \sigma_3) + (\sigma_2 \circ_h \sigma_3))(f) = \\ (\sigma_2 \circ_h \sigma_3)(f)((\sigma_1 \circ_h \sigma_3)(f), (\sigma_1 \circ_h \sigma_3)(f)) \\ = f(f(y, y), x)(f(x, f(y, y)), f(x, f(y, y))) \\ = f(f(f(x, f(y, y)), f(x, f(y, y))), f(x, f(y, y))).\end{gathered}$$

On the set $Hyp(\tau)$ not only operations, but also relations can be defined. Let $\sigma_1, \sigma_2 \in Hyp(\tau)$. Then we define $\sigma_1 \preceq_{\mathcal{R}} \sigma_2$ if and only if there is a hypersubstitution σ such that $\sigma_1 = \sigma_2 \circ_h \sigma$. Since $Hyp(\tau)$ is a monoid, $\preceq_{\mathcal{R}}$ is reflexive and transitive, i.e. a quasiorder.
Similarly, we define $\sigma_1 \preceq_{\mathcal{L}} \sigma_2$ if and only if there is a hypersubstitution σ such that $\sigma_1 = \sigma \circ_h \sigma_2$. The relation $\preceq_{\mathcal{L}}$ is also a quasiorder. Then it is easy to see (and well-known) that $\mathcal{R} = \preceq_{\mathcal{R}} \cap \preceq_{\mathcal{R}}^{-1}$ and $\mathcal{L} = \preceq_{\mathcal{L}} \cap \preceq_{\mathcal{L}}^{-1}$ are equivalence relations and are called *Green's relations* $\mathcal{R}$ and $\mathcal{L}$. The relations $\preceq_{\mathcal{L}}$ and $\preceq_{\mathcal{R}}$ induce partial order relations on the quotient sets $Hyp(\tau)/\mathcal{R}$ and $Hyp(\tau)/\mathcal{L}$, respectively.
Green's relations $\mathcal{H}$ and $\mathcal{D}$ are defined by $\mathcal{H} := \mathcal{R} \cap \mathcal{L}$ and $\mathcal{D} := \mathcal{R} \circ \mathcal{L}$ and $\mathcal{J}$ is defined by

$$\sigma_1 \mathcal{J} \sigma_2 :\Leftrightarrow \exists \sigma, \sigma', \gamma, \gamma' \in Hyp(\tau)\ (\sigma_1 = \sigma \circ_h \sigma_2 \circ_h \sigma', \sigma_2 = \gamma \circ_h \sigma_1 \circ_h \gamma').$$

We recall the following properties of Green's relations $\mathcal{R}$ and $\mathcal{L}$:

Proposition 1.1. *([7])*

(i) $\mathcal{R} \circ \mathcal{L} = \mathcal{L} \circ \mathcal{R}$,

(ii) $\mathcal{R}$ *is a left congruence on* $Hyp(\tau)$,

(iii) $\mathcal{L}$ *is a right congruence on* $Hyp(\tau)$.

In [5] we considered special submonoids of $Hyp(\tau)$. The images of the hypersubstitutions from these submonoids are called *full terms* and *strongly full terms* and are inductively defined as follows:

Definition 1.2. *Let* $f_i, i \in I$, *be an* n_i*-ary operation symbol and let* $s : \{1, \ldots, n_i\} \to \{1, \ldots, n_i\}$ *be a permutation, then*

(i) $f_i(x_{s(1)}, \ldots, x_{s(n_i)})$ *is a full term and*

(ii) if $f_j, j \in J$, *is an* n_j*-ary operation symbol and if* $t_1, \ldots, t_{n_j}$ *are full terms, then* $f_j(t_1, \ldots, t_{n_j})$ *is a full term.*

Let $W_\tau^f(X)$ *be the set of all full terms of type* τ.

Strongly full terms are defined as follows:

Definition 1.3. *(i) For every* n_i*-ary operation symbol* $f_i, i \in I$ *the term* $f_i(x_1, \ldots, x_{n_i})$ *is strongly full,*

(ii) if $t_1, \ldots, t_{n_i}$ *are strongly full and if* $f_i, i \in I$, *is an* n_i*-ary operation symbol, then* $f_i(t_1, \ldots, t_{n_i})$ *is strongly full.*

We denote by $W_\tau^{sf}(X)$ *the set of all strongly full terms of type* τ.

Definition 1.4. *A hypersubstitution* σ *is called full if* $\sigma(f_i) \in W_\tau^f(X_{n_i})$ *for all* $i \in I$ *and strongly full if for all* $i \in I$ *the images of the operation symbols* f_i *belong to* $W_\tau^{sf}(X_{n_i})$. *By* $Hyp^f(\tau)$ *and* $Hyp^{sf}(\tau)$, *respectively, we denote the set of all full and the set of all strongly full hypersubstitutions of type* τ.

Then in [5] it was proved:

Lemma 1.5. *The sets* $Hyp^f(\tau)$ *and* $Hyp^{sf}(\tau)$ *form submonoids of the monoid* $Hyp(\tau)$.

2. Green's Relations and the Complexity of a Hypersubstitutions

There are different ways to measure the complexity of a term. If $vb(t)$ denotes the number of variables occurring in the term t, then vb is an example for a complexity measure of terms. The most commonly used measurement of the complexity of terms of type τ is the usual depth of a term which is defined inductively by the following steps:

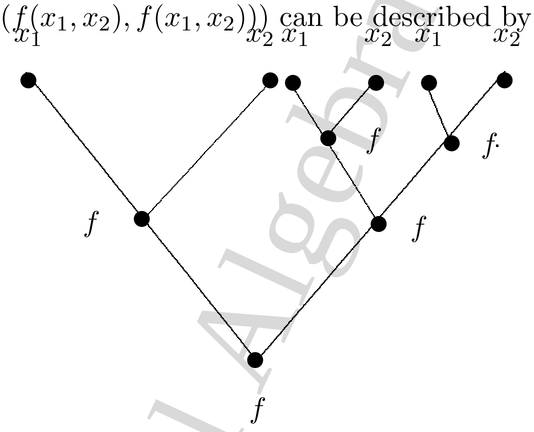
(i) If $t = x \in X$, then

$$depth(t) = 0.$$

(ii) If $t = f_i(t_1, \ldots, t_{n_i})$, then

$$depth(t) = max\{depth(t_1), \ldots, depth(t_{n_i})\} + 1.$$

To decribe some more complexity functions we often identify a term with the tree diagram used to represent it. For instance, the term $f(f(x_1, x_2), f(f(x_1, x_2), f(x_1, x_2)))$ can be described by the diagram:



Then the depth of a term t is the length of the longest path from the root to a vertex labelled by a variable (a leaf) in the tree. Instead of $depth(t)$ one can also speak of the $maxdepth(t)$. The minimum depth of a term t, denoted by $mindepth(t)$, is the length of the shortest path from the root to a leaf in the tree and is defined inductively by

(i) If t is a variable,then

$$mindepth(t) = 0.$$

(ii) If $t = f_i(t_1, \ldots, t_{n_i})$,then

$$mindepth(t) = min\{mindepth(t_1), \ldots, mindepth(t_{n_i})\} + 1.$$

Using the depth and the mindepth we can define another kind of terms.

Definition 2.1. *Let $t \in W_\tau(X_n)$ be an n-ary term of type τ. Then t is called path-regular (for short a pr-term) if $mindepth(t) = depth(t)$. Let $W_\tau^{pr}(X_n)$ be the set of all n-ary pr-terms of type τ. A hypersubstitution is called path-regular, if the terms $\sigma(f_i)$ are path-regular for every $i \in I$. Let $Hyp^{pr}(\tau)$ be the set of all path-regular hypersubstitutions.*

In [5] it was proved:

Proposition 2.2. *$Hyp^{pr}(\tau)$ forms a submonoid of $Hyp(\tau)$.*

All these complexity measures are particular cases of the following valuation of terms:

Definition 2.3. *Let $\mathcal{F}_\tau(X) = (W_\tau(X); (\bar{f}_i)_{i\in I})$ with $\bar{f}_i : \{t_1, \ldots, t_{n_i}\} \mapsto f_i(t_1, \ldots, t_{n_i})$ be the absolutely free term algebra of type τ on a countable set X, and let $\mathbb{N}_\tau = (\mathbb{N}; (f_i^{\mathbb{N}})_{i\in I})$ be an algebra of type τ defined on the set of all natural numbers. Then a mapping $v : X \to \mathbb{N}_\tau$ is called a valuation of terms of type τ into $\mathbb{N}_\tau$ if the following conditions are satisfied:*

(i) $v(x) = a$ if $x \in X$ and $a \in \mathbb{N}$.

(ii) $v(t) \geq v(x)$ for every variable x and every term t (see [6]).

From the freeness of $\mathcal{F}_\tau(X)$ we obtain a uniquely determined homomorphism $\hat{v} : \mathcal{F}_\tau(X) \to \mathbb{N}_\tau$ which extends v. For short, we denote this homomorphism also by v and will call it valuation of terms. In the case of $depth(t)$ the operations $f_i^{\mathbb{N}}$ are defined by $f_i^{\mathbb{N}}(a_1, \ldots, a_{n_i}) = max\{a_1, \ldots, a_{n_i}\} + 1$ and for $mindepth(t)$ we have $f_i^{\mathbb{N}}(a_1, \ldots, a_{n_i}) = min\{a_1, \ldots, a_{n_i}\} + 1$. Both kinds of operations are monotone with respect to the usual order $\leq$ on $\mathbb{N}$.

So, in many case the mapping v satisfies the following condition

(OC) If $a_j \leq b_j$ for $1 \leq j \leq n_i$ and f_i is an n_i-ary operation symbol of type τ then for the corresponding operations $f_i^{\mathbb{N}}$ we have $f_i^{\mathbb{N}}(a_1, \ldots, a_{n_i}) \leq f_i^{\mathbb{N}}(b_1, \ldots, b_{n_i})$ Here we denote by $\leq$ the usual order on the set of natural numbers.

For more background on valuation of terms see [6].

This can be applied to hypersubstitutions as follows:

Definition 2.4. *Let σ be a hypersubstitution of type τ. Then*

$$\begin{array}{lll} depth(\sigma) & := & min\{depth(\sigma(f_i)) \mid i \in I\} \\ v(\sigma) & := & min\{v(\sigma(f_i)) \mid i \in I\}. \end{array}$$

For the type $\tau = (n)$ and for full terms in [4] it was proved:

Proposition 2.5. *Let $\tau = (n), n \geq 1$ and let $t \in W_\tau^f(X)$ be a full term. Then*

$$depth(\hat{\sigma}[t]) = depth(\sigma(f))depth(t).$$

As a consequence we obtain:

Proposition 2.6. *Assume that $\tau = (n)$. The mapping $depth : Hyp^f(n) \to \mathbb{N}^+$ defined by $\sigma \mapsto depth(\sigma)$ is a homomorphism of $(Hyp^f(n); \circ_h, \sigma_{id})$ onto the monoid $(\mathbb{N}^+; \cdot, 1)$ of all positive integers.*

Proof. The mapping *depth* is well-defined and for every natural number $n \geq 1$ there is a full hypersubstitution σ with $depth(\sigma) = n$. Therefore *depth* is surjective. Further we have

$$\begin{aligned} depth(\sigma_1 \circ_h \sigma_2) &= depth((\sigma_1 \circ_h \sigma_2)(f)) = depth(\hat{\sigma_1}[\sigma_2(f)]) \\ &= depth(\sigma_1(f))depth(\sigma_2(f)) = depth(\sigma_1)depth(\sigma_2) \end{aligned}$$

by Proposition 2.5 and

$$depth(\sigma_{id}) = depth(\sigma_{id}(f)) = depth(f(x_1, \ldots, x_n)) = 1.$$

□

By the homomorphism theorem $(\mathbb{N}^+; \cdot, 1)$ is isomorphic to the quotient monoid $Hyp^f(n)/kerdepth$ with $kerdepth = \{(\sigma_1, \sigma_2) \mid \sigma_1, \sigma_2 \in Hyp^f(n), depth(\sigma_1) = depth(\sigma_2)\}$.

Proposition 2.5 has some consequences for Green's relations. First of all, if $\sigma_1 = \sigma_2 \circ_h \sigma_3$, then by Proposition 2.5 $depth(\sigma_2)$ divides $depth(\sigma_1)$ and $depth(\sigma_3)$ divides $depth(\sigma_1)$. One more consequence of Proposition 2.5 is that the monoid $(Hyp^f(n); \circ_h, \sigma_{id})$ is not finitely generated. The homomorphism *depth* maps any generating system of $Hyp^f(n)$ onto a generating system of $(\mathbb{N}^+; \cdot, 1)$. Every generating system of $(\mathbb{N}; \cdot, 1)$ contains the infinite set of all prime numbers. This shows that there is no finite generating system of $(Hyp^f(n); \circ_h, \sigma_{id})$.
The homomorphism *depth* preserves also the quasiorders $\preceq_\mathcal{R}$ and $\preceq_\mathcal{L}$ on $Hyp(\tau)$ since

$$\begin{aligned} &\sigma_1 \preceq_\mathcal{R} \sigma_2 \Rightarrow \exists \rho \in Hyp(\tau)(\sigma_1 = \sigma_2 \circ_h \rho) \Rightarrow \\ &\qquad \Rightarrow depth(\sigma_2) | depth(\sigma_1) \Rightarrow depth(\sigma_2) \leq depth(\sigma_1) \end{aligned}$$

(Here $\leq$ denotes the usual order on $\mathbb{N}$).

Corollary 2.7. *Let $\sigma_1, \sigma_2 \in Hyp^f(n)$ and let $\mathcal{K} \in \{\mathcal{R}, \mathcal{L}, \mathcal{H}, \mathcal{D}, \mathcal{J}\}$, then $\sigma_1 \mathcal{K} \sigma_2$ implies $depth(\sigma_1) = depth(\sigma_2)$.*

Proof. For $\mathcal{K} = \mathcal{R}$ we have

$$\begin{aligned} \sigma_1 \mathcal{R} \sigma_2 &\Longrightarrow \exists \rho, \rho' \in Hyp(\tau)\ (\sigma_1 = \sigma_2 \circ_h \rho \wedge \sigma_2 = \sigma_1 \circ_h \rho') \\ &\Longrightarrow depth(\sigma_2) \leq depth(\sigma_1) \wedge depth(\sigma_1) \leq depth(\sigma_2) \\ &\Longrightarrow depth(\sigma_1) = depth(\sigma_2). \end{aligned}$$

For the relation $\mathcal{L}$ we conclude in a similar way.
For $\mathcal{H}$ we have:

$$\sigma_1 \mathcal{H} \sigma_2 \Longrightarrow \sigma_1 (\mathcal{R} \cap \mathcal{L}) \sigma_2 \Longrightarrow \sigma_1 \mathcal{R} \sigma_2 \wedge \sigma_1 \mathcal{L} \sigma_2 \Longrightarrow depth(\sigma_1) = depth(\sigma_2).$$

Considering $\mathcal{D}$ we obtain:

$$\begin{aligned} \sigma_1 \mathcal{D} \sigma_2 \Longrightarrow \sigma_1 (\mathcal{R} \circ \mathcal{L}) \sigma_2 \Longrightarrow \exists \sigma \in Hyp(\tau) (\sigma_1 \mathcal{L} \sigma \wedge \sigma \mathcal{R} \sigma_2) \\ \Longrightarrow depth(\sigma_1) = depth(\sigma) \wedge depth(\sigma) = depth(\sigma_2). \end{aligned}$$

Finally, for $\mathcal{J}$ we get

$$\begin{aligned} \sigma_1 \mathcal{J} \sigma_2 \Rightarrow \exists \sigma, \sigma', \gamma, \gamma' \in Hyp(\tau)\ (\sigma_1 = \sigma \circ_h \sigma_2 \circ_h \sigma' \wedge \sigma_2 = \gamma \circ_h \sigma_1 \circ_h \gamma') \\ \Rightarrow depth(\sigma_1) | depth(\sigma_2) \wedge depth(\sigma_2) | depth(\sigma_1) \\ \Rightarrow depth(\sigma_1) = depth(\sigma_2). \end{aligned}$$

□

For the depth of the hypersubstitutions $\rho, \rho', \gamma, \gamma'$ which are needed in the definitions of $\mathcal{R}$, $\mathcal{L}$ and $\mathcal{J}$ we have $depth(\rho) = depth(\rho') = depth(\gamma) = depth(\gamma') = 1$.

Further we have

Corollary 2.8. *$\sigma \in Hyp^f(n)$ is invertible if and only if $depth(\sigma) = 1$ and idempotent if and only if $\sigma = \sigma_{id}$.*

Proof. If σ is invertible, then there exists a hypersubstitution σ' such that $\sigma \circ \sigma' = \sigma' \circ \sigma = \sigma_{id}$. Now from Proposition 2.5 we obtain $depth(\sigma) \cdot depth(\sigma') = 1$ and then $depth(\sigma) = 1$. If conversely $depth(\sigma) = 1$, then there is a permutation s on the set $\{1, \ldots, n\}$ such that $\sigma(f) = f(x_{s(1)}, \ldots, x_{s(n)})$, but then the hypersubstitution σ' with $\sigma'(f) = f(x_{s^{-1}(1)}, \ldots, x_{s^{-1}(n)})$ satisfies

$$\begin{aligned} (\sigma \circ_h \sigma')(f) = f(x_{(s \circ s^{-1})(1)}, \ldots, x_{(s \circ s^{-1})(n)}) = f(x_1, \ldots, x_n) = \\ = \sigma_{id}(f) = f(x_{(s^{-1} \circ s)(1)}, \ldots, x_{(s^{-1} \circ s)(n)}) = (\sigma' \circ \sigma)(f). \end{aligned}$$

Therefore σ is invertible. The second proposition follows from

$$\sigma^2 = \sigma \Rightarrow depth(\sigma^2) = depth(\sigma) \Rightarrow depth(\sigma) = 1 \Rightarrow \\ \Rightarrow \exists s \in S_n(\sigma(f) = f(x_{s(1)}, \ldots, x_{s(n)})) \Rightarrow s = \sigma_{id}.$$

□

Clearly, the set of all invertible elements of $(Hyp^f(n); \circ_h, \sigma_{id})$ is the maximal subgroup of $(Hyp^f(n); \circ_h, \sigma_{id})$ and is isomorphic to the full symmetric group S_n of all permutations on $\{1, \ldots, n\}$.

Assume that σ_1, σ_2 are full hypersubstitutions. In [4] it was proved that the superposition of full terms is full. Therefore $\sigma_1 + \sigma_2$ is a full hypersubstitution and we have:

Theorem 2.9. $(Hyp^f(n); \circ_h, +, \sigma_{id})$ *is a left-seminearring with identity and the function depth is a homomorphism onto the semiring* $(\mathbb{N}^+; \cdot, +, 1)$ *of natural numbers with identity.*

Proof. We proved already that all defining identities of a left-seminearring are satisfied. By Definition 2.4 we have

$$\begin{aligned} depth(\sigma_1 + \sigma_2) &= depth(\sigma_1 + \sigma_2)(f) \\ &= depth(\sigma_2(f)(\sigma_1(f), \ldots, \sigma_1(f))) \\ &= depth(\sigma_1) + depth(\sigma_2). \end{aligned}$$

The rest follows from Proposition 2.5. □

Seminearrings were considered in [8] and [9].

Further we have

Proposition 2.10. *The structures* $(Hyp^{sf}(n); \circ_h, +, \sigma_{id})$ *and* $(Hyp^{pr}(n); \circ_h, +, \sigma_{id})$ *are left-seminearrings and* $(Hyp^{sf}(n); \circ_h, +, \sigma_{id})$ *is a sub-left-seminearring of the left-seminearring* $(Hyp^f(n); \circ_h, +, \sigma_{id})$.

Proof. $(Hyp^{sf}(n); \circ_h, \sigma_{id})$ is a submonoid of $(Hyp^f(n); \circ_h, +, \sigma_{id})$. Assume that $\sigma_1, \sigma_2 \in Hyp^f(n)$. Then $\sigma_2(f)(\sigma_1(f), \ldots, \sigma_1(f))$. If we substitute for x_1 the strongly full term $\sigma_1(f)$, etc., and finally for x_n the strongly full term $\sigma_1(f)$, then by the inductive definition of strongly full terms the resulting term $\sigma_2(f)(\sigma_1(f)), \ldots, \sigma_1(f))$ is strongly full and $\sigma_1 + \sigma_2 \in Hyp^{sf}(n)$.
Assume now that $\sigma_1, \sigma_2 \in Hyp^{pr}(n)$, i.e.

$$mindepth(\sigma_2(f)) = depth(\sigma_j(f)), j = 1, 2.$$

Then we have also

$$\begin{aligned} mindepth((\sigma_1 + \sigma_2)(f)) &= mindepth(\sigma_2(f)(\sigma_1(f), \ldots, \sigma_1(f))) \\ &= mindepth(\sigma_1(f)) + mindepth(\sigma_2(f)) \end{aligned}$$

and thus

$$mindepth(\sigma_1 + \sigma_2) = mindepth(\sigma_1) + mindepth(\sigma_2) = \\ = depth(\sigma_1) + depth(\sigma_2) = depth(\sigma_1 + \sigma_2)$$

and then

$$\sigma_1 + \sigma_2 \in Hyp^{pr}(n).$$

□

Further we have

Proposition 2.11. *$(Hyp^{sf}(n); \circ_h, +) \cap (Hyp^{pr}(n); \circ_h, +)$ is the left-seminearring generated by σ_{id}.*

Proof. The one-element set σ_{id} is closed under the multiplication. Therefore, since the addition of hypersubstitutions is associative, every element of the left-seminearring generated by σ_{id} can be written as $n\sigma_{id}$ for some natural number n. We show by induction on n that every hypersubstitution of the form $n\sigma_{id}$ belongs to $Hyp^{sf}(n)$ and to $Hyp^{pr}(n)$ and therefore to the intersection. This is clear for σ_{id}. Assume that $n\sigma_{id} \in Hyp^{sf}(n) \cap Hyp^{pr}(n)$, then $(n+1)\sigma_{id}(f) = n\sigma_{id} + \sigma_{id}(f) = \sigma_{id}(f)(n\sigma_{id}(f), \ldots, n\sigma_{id}(f)) = f(n\sigma_{id}(f), \ldots, n\sigma_{id}(f))$ and by the definition of strongly full and of path-regular hypersubstitutions we have $(n+1)\sigma_{id} \in Hyp^{sf}(n) \cap Hyp^{pr}(n)$.
Conversely, assume that $\sigma \in Hyp^{sf}(n) \cap Hyp^{pr}(n)$. Since σ is full, there are terms $t_1, \ldots, t_n$ such that $\sigma(f) = f(t_1, \ldots, t_n)$. We give a proof by $depth(\sigma)$. If $depth(\sigma) = 1$, i.e. $depth(\sigma(f)) = 1$, then $\sigma(f) = f(x_1, \ldots, x_n) = \sigma_{id}(f)$ and thus $\sigma \in \langle\sigma_{id}\rangle$. Assume that every σ with $depth(\sigma) = n$ belongs to $\langle\sigma_{id}\rangle$ and let σ' be a hypersubstitution with $depth(\sigma') = n+1$. Thus $\sigma'(f) = f(t_1, \ldots, t_n)$ where $t_1, \ldots, t_n$ are full and path-regular terms. Consider the hypersubstitutions $\sigma_1, \ldots, \sigma_n$ with $\sigma_i(f) = t_i$. Since $depth(\sigma_i) = n$, we have $\sigma_i \in \langle\sigma_{id}\rangle$. If there numbers n_i such that $\sigma_j = n_i\sigma_{id}, n_i \neq n_j$ for $i \neq j$, then

$$min\{mindepth(n_1\sigma_{id}), \ldots, mindepth(n_n\sigma_{id})\} = \\ = max\{depth(n_1\sigma_{id}), \ldots, depth(n_n\sigma_{id})\}.$$

Therefore for every $i = 1, \ldots, n$ we have $\sigma_i = n\sigma_{id}$ and therefore $\sigma' = (n+1)\sigma_{id} \in \langle\sigma_{id}\rangle$. This shows that $\langle\sigma_{id}\rangle = Hyp^{sf}(n) \cap Hyp^{pr}(n)$. □

We remark that the set of all hypersubstitutions of arbitrary type τ is also closed under our addition and is called a left-seminearring since the proof of the associativity and left distributivity did not use the type

(n). Then a consequence of Proposition 2.10 is that the left- seminearring $(Hyp^f(n); \circ_h, +, \sigma_{id})$ has no finite sub-left-seminearring and that every left-seminearring of hypersubstitutions contains the infinite left-seminearring $\langle\sigma_{id}\rangle = Hyp^{sf}(n)$.

Remark further that the mapping $n\sigma_{id} \mapsto n$ defines an isomorphism between $\langle\sigma_{id}\rangle$ and $(\mathbb{N}; +, \cdot, 1)$. This show that $\langle\sigma_{id}\rangle$ is a semiring with cancellation rules for both operations, and with commutative addition. Furthermore $\langle\sigma_{id}\rangle$ has no zero-divisors. Now we want to generalize Corollary 2.7 to the valuation of terms of type τ into $\mathcal{N}_\tau$ introduced in Definition 2.3.

We mention the following Fact which was proved in [6]:

Fact: Let v be a valuation of terms into $\mathcal{N}_\tau$ which satisfies the condition (OC). Then for any n-ary terms $t_1, \ldots, t_m$ and for an arbitrary m-ary term t we have

$$v(t(t_1, \ldots, t_m)) \geq v(t).$$

Now we prove:

Proposition 2.12. *Let $\sigma_1, \sigma_2 \in Hyp^f(n)$. If $\sigma_1 \mathcal{R} \sigma_2$, then for every valuation which satisfies (OC) we have $v(\sigma_1) = v(\sigma_2)$.*

Proof. If $\sigma_1 \mathcal{R} \sigma_2$, then there exist hypersubstitutions $\sigma, \sigma' \in Hyp^f(n)$ such that $\sigma_1 = \sigma_2 \circ_h \sigma$ and $\sigma_2 = \sigma_1 \circ_h \sigma'$ and therefore from $\sigma_1(f) = (\sigma_2 \circ_h \sigma)(f)$ follows $\sigma_1(f) = \hat{\sigma}_2[\sigma(f)]$ and $\sigma_2(f) = \hat{\sigma}_1[\sigma'(f)]$. Since $\tau = n$ and since $\sigma \in Hyp^f(n)$, the term $\sigma(f)$ has the form $f(t_1, \ldots, t_n)$ and then $\hat{\sigma}_2[\sigma(f)]$ has the form $\sigma_2(f)(\hat{\sigma}_2[t_1], \ldots, \hat{\sigma}_2[t_n])$. Applying the Fact we see that $v((\sigma_2 \circ \sigma)(f)) \geq v(\sigma_2(f))$ and then $v(\sigma_1) \geq v(\sigma_2)$. Using $\sigma_2 = \sigma_1 \circ_h \sigma'$ we get $v(\sigma_2) \geq v(\sigma_1)$. Altogether, we have $v(\sigma_1) = v(\sigma_2)$. ☐

Clearly, if $\sigma_1 \mathcal{H} \sigma_2$ and if v satisfies (OC) we have also $v(\sigma_1) = v(\sigma_2)$. Because of $(\sigma_1 + \sigma_2)(f) = \sigma_2(f)(\sigma_1(f), \ldots, \sigma_1(f))$ from the fact follows that $v(\sigma_1 + \sigma_2) \leq v(\sigma_2)$, while $v(\sigma_1 \circ_h \sigma_2) \leq v(\sigma_1)$.

3. A Characterization of Green's relation $\mathcal{R}$

The condition $depth(\sigma) = depth(\sigma')$ turns out to be necessary, but not sufficient for $\sigma_1 \mathcal{R} \sigma_2$. Indeed, if $\sigma_1, \sigma_2 \in Hyp^f(2)$ and $\sigma_1 \neq \sigma_2$ then $\sigma_1 \mathcal{R} \sigma_2$ implies $\sigma_1 = \sigma_2 \circ_h \sigma_{x_1x_2}$ or $\sigma_2 = \sigma_1 \circ_h \sigma_{x_2x_1}$. For instance, for type $\tau = (2)$ the hypersubstitutions σ_1 with

$$\sigma_1(f) = f(f(x_1, x_2), f(f(x_1, x_2), f(x_2, x_1)))$$

and σ_2 with

$$\sigma_2(f) = f(f(f(x_1, x_2), f(x_2, x_1)), f(x_1, x_2))$$

satisfy $depth(\sigma_1) = depth(\sigma_2)$, but σ_1 and σ_2 are not $\mathcal{R}$-related. Therefore we need some more conditions.

For any n-ary term t we denote by $S_n^{n,x}(t)$ the term arising from t if we substitute for each variable the variable x.

Then we get:

Proposition 3.1. *Assume that $\sigma_1, \sigma_2 \in Hyp^f(n)$. If $\sigma_1 \mathcal{R} \sigma_2$, then*

$$S_n^{n,x}(\sigma_1(f)) = S_n^{n,x}(\sigma_2(f)).$$

Proof. If $\sigma_1 \mathcal{R} \sigma_2$ then there are hypersubstitutions σ and σ' such that $\sigma_1 = \sigma_2 \circ_h \sigma$ and $\sigma_2 = \sigma_1 \circ_h \sigma'$ and then $\sigma_1 = \sigma_1 \circ_h (\sigma \circ_h \sigma')$. By Proposition 2.5 we get $depth(\sigma_1) = depth(\sigma_1) depth(\sigma) depth(\sigma')$. From this we obtain $depth(\sigma) = depth(\sigma') = 1$. Since σ and σ' are full hypersubstitutions, we have $\sigma(f) = f(x_{s(1)}, \ldots, x_{s(n)})$, $\sigma'(f) = f(x_{s'(1)}, \ldots, x_{s'(n)})$ for permutations s,s' on $\{1, \ldots, n\}$. From this there follows

$$\begin{aligned} S_n^{n,x}(\sigma_1(f)) &= S_n^{n,x}((\sigma_2 \circ_h \sigma)(f)) \\ &= S_n^{n,x}(\hat{\sigma_2}[\sigma(f)]) \\ &= S_n^{n,x}(\hat{\sigma_2}[f(x_{s(1)}, \ldots, x_{s(n)})]) \\ &= S_n^{n,x}(\hat{\sigma_2}[f(x_{s'(1)}, \ldots, x_{s'(n)})]) \\ &= S_n^{n,x}(\sigma_2(f)). \end{aligned}$$

□

Every full term $t \in W_n^f(X_n)$ contains subterms of the form

$$f(x_{s_j(1)}, \ldots, x_{s_j(n)})$$

for permutations s_j on $\{1, \ldots, n\}$. Considering the tree of the term t by $P(t)$ we denote the sequence of all permutations on $\{1, \ldots, n\}$ which are needed in t, written from the left to the right, i.e.,

$P(t) := \{(s_1, \ldots, s_m) \mid f(x_{s_i(1)}, \ldots, x_{s_i(n)})$ is a subterm of t for $1 \leq i \leq m$ and where these subterms are ordered in the tree of t from the left to the right$\}$.

Example: Consider for $\tau = (3)$ the term

$$t = f(f(x_2, x_1, x_3), f(f(x_1, x_2, x_3), f(x_2, x_1, x_3), f(x_3, x_2, x_1)), f(x_3, x_1, x_2)).$$

Then $P(t) = ((12), (1), (12), (13), (132))$ if we write the permutations which are needed as cycles. Clearly, two terms $t_1, t_2 \in W_{(n)}^f(X_n)$ are equal if and only if $S_n^{n,x}(t_1) = S_n^{n,x}(t_2)$ and $P(t_1) = P(t_2)$.

Now we have

Proposition 3.2. *Let $\sigma_1, \sigma_2 \in Hyp^f(n)$ and assume that $P(\sigma_1(f)) = (u_1, \ldots, u_m)$ and $P(\sigma_2(f)) = (v_1, \ldots, v_l)$. If $\sigma_1 \mathcal{R} \sigma_2$ then $m = l$ and $u_1 v_1^{-1} = \cdots = u_m v_m^{-1}$.*

Proof. Assume that $\sigma_1 \mathcal{R} \sigma_2$. There are hypersubstitutions σ, σ' such that $\sigma_1 = \sigma_2 \circ_h \sigma$ and $\sigma_2 = \sigma_1 \circ_h \sigma'$ and therefore $depth(\sigma) = depth(\sigma') = 1$. It follows that there are permutations $s, s' : \{1, \ldots, n\} \to \{1, \ldots, n\}$ such that $\sigma_1(f) = \hat{\sigma_2}[f(x_{s(1)}, \ldots, x_{s(n)})]$ and $\sigma_2(f) = \hat{\sigma_1}[f(x_{s'(1)}, \ldots, x_{s'(n)})]$. Then

$$P(\sigma_1(f)) = P(\hat{\sigma_2}[f(x_{s(1)}, \ldots, x_{s(n)}]) = \\ = P(\sigma_2(f)(x_{s(1)}, \ldots, x_{s(n)})) = (s \circ v_1, \ldots, s \circ v_l).$$

Similarly, we get

$$P(\sigma_2(f)) = P(\hat{\sigma_1}[f(x_{s'(1)}, \ldots, x_{s'(n)}]) = \\ = P(\sigma_1(f)(x_{s'(1)}, \ldots, x_{s'(n)})) = (s' \circ u_1, \ldots, s' \circ u_m).$$

By Proposition 3.1 from $\sigma_1 \mathcal{R} \sigma_2$ there follows $S_n^{n,x}(\sigma_1(f)) = S_n^{n,x}(\sigma_2(f))$. Since the trees of $S_n^{n,x}(\sigma_1(f))$ and $\sigma_1(f)$ differ only in the labeling of the leaves, the structure of the tree of $\sigma_1(f)$ and of $\sigma_2(f)$ is equal and therefore the number of permutations s occurring in $\sigma_1(f)$ and $\sigma_2(f)$ is equal, i.e. we have $m = l$ and then $(s \circ v_1, \ldots, s \circ v_m) = (s' \circ u_1, \ldots, s' \circ u_m)$ implies $s \circ v_j = s' \circ u_j$ for every $1 \leq j \leq m$. From this equation we obtain $u_j \circ v_j^{-1} = s \circ s'^{-1}$ for every $1 \leq j \leq m$ and this means $u_1 \circ v_1^{-1} = \cdots = u_m \circ v_m^{-1}$. □

It turns out that both conditions, Proposition 3.1 and Proposition 3.2 together, characterize Green's relation $\mathcal{R}$.

Theorem 3.3. *Let $\sigma_1, \sigma_2 \in Hyp^f(n)$. Then the following conditions are equivalent:*

(i) $\sigma_1 \mathcal{R} \sigma_2$,

(ii) $S_n^{n,x}(\sigma_1(f)) = S_n^{n,x}(\sigma_2(f))$ *and* $u_1 \circ v_1^{-1} = \cdots = u_m \circ v_m^{-1}$ *where* $P(\sigma_1(f)) = (u_1, \ldots, u_m)$ *and* $P(\sigma_2(f)) = (v_1, \ldots, v_m)$.

Proof. (*i*) ⇒ (*ii*) was already proved.

(*ii*) ⇒ (*i*): We form $s = u_1 \circ v_1^{-1}, s^{-1} = v_1 \circ u_1^{-1}$ and consider $\sigma, \sigma' \in Hyp^f(n)$ defined by $\sigma(f) = f(x_{s(1)}, \ldots, x_{s(n)})$ and $\sigma'(f) = f(x_{s^{-1}(1)}, \ldots, x_{s^{-1}(n)})$. Clearly, $\sigma' = \sigma^{-1}$. Now we prove that $\sigma_1 = \sigma_2 \circ_h \sigma$ using the remark before Proposition 3.2 and showing that $S_n^{n,x}(\sigma_1(f)) = S_n^{n,x}((\sigma_2 \circ \sigma)(f))$ and $P(\sigma_1(f)) = P((\sigma_2 \circ_h \sigma)(f))$. Indeed, we have

$$S_n^{n,x}((\sigma_2 \circ \sigma)(f)) = S_n^{n,x}(\hat{\sigma_2}[\sigma(f)]) = S_n^{n,x}(\hat{\sigma_2}[f(x_{s(1)}, \ldots, x_{s(n)})]) = \\ = S_n^{n,x}(\sigma_2(f)(x_{s(1)}, \ldots, x_{s(n)})) = S_n^{n,x}(\sigma_2(f)).$$

Further $P((\sigma_2 \circ_h \sigma)(f)) = P(\sigma_2(f)(x_{s(1)}, \ldots, x_{s(n)})) = (s \circ v_1, \ldots, s \circ v_m) = (u_1, \ldots, u_m) = P(\sigma_1(f))$. Since the inverse of σ exists, we get $\sigma_2 = \sigma_1 \circ \sigma^{-1}$ and altogether we have $\sigma_1 R \sigma_2$. □

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