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On some Leibniz algebras, having small dimension

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Abstract. The first step in the study of all types of algebras is the description of such algebras having small dimensions. The structure of 3-dimensional Leibniz algebras is more complicated than 1 and 2-dimensional cases. In this paper, we consider the structure of Leibniz algebras of dimension 3 over the finite fields. In some cases, the structure of the algebra essentially depends on the characteristic of the field, in others on the solvability of specific equations in the field, and so on.

Introduction

Let L be an algebra over finite field F with the binary operations $+$ and $[\cdot,\cdot]$. Then L is called a *Leibniz algebra* (more precisely a *left Leibniz algebra*) if it satisfies the (left) Leibniz identity

$$[[a,b],c] = [a,[b,c]] - [b,[a,c]]$$

for all $a,b,c \in L$.

We will also use another form of this identity:

$$[a,[b,c]] = [[a,b],c] + [b,[a,c]].$$

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Leibniz algebras appeared first in the papers of A.M. Bloh [5–7], in which he called them the *D-algebras*. However, in that time these works were not in demand, and they have not been properly developed. Only after two decades, a real interest to Leibniz algebras rose. It happened thanks to the work of J.-L. Loday [15] (see also [16, Section 10.6]), who "rediscovered" these algebras and used the term *Leibniz algebras* since it was Gottfried Wilhelm Leibniz who discovered and proved the *Leibniz rule* for differentiation of functions.

Let L be a Leibniz algebra over finite field F. If A, B are subspaces of L, then $[A, B]$ will denote a subspace generated by all elements $[a, b]$ where $a \in A$, $b \in B$. As usual, a subspace A of L is called a *subalgebra* of L, if $[x, y] \in A$ for every $x, y \in A$. It follows that $[A, A] \leqslant A$.

Let L be a Leibniz algebra over finite field F, M be non-empty subset of L, then $\langle M \rangle$ denote the subalgebra of L generated by M.

A subalgebra A of L is called a *left* (respectively *right*) *ideal* of L, if $[y, x] \in A$ (respectively $[x, y] \in A$) for every $x \in A$, $y \in L$. In other words, if A is a left (respectively right) ideal, then $[L, A] \leqslant A$ (respectively $[A, L] \leqslant A$).

A subalgebra A of L is called an *ideal* of L (more precisely, *two-sided ideal*) if it is both a left ideal and a right ideal, that is $[y, x], [x, y] \in A$ for every $x \in A$, $y \in L$.

If A is an ideal of L, we can consider a *factor-algebra* L/A. It is not hard to see that this factor-algebra also is a Leibniz algebra.

As usual, a Leibniz algebra L is called *abelian* if $[x, y] = 0$ for all elements $x, y \in L$. In abelian Leibniz algebra every subspace is a subalgebra and an ideal.

Denote by $\mathbf{Leib}(L)$ the subspace, generated by the elements $[a, a]$, $a \in L$. It is possible to prove that $\mathbf{Leib}(L)$ is an ideal of L such that $L/\mathbf{Leib}(L)$ is a Lie algebra. Conversely, if H is an ideal of L such that L/H is a Lie algebra, then $\mathbf{Leib}(L) \leqslant H$.

The ideal $\mathbf{Leib}(L)$ is called a *Leibniz kernel* of algebra L.

We note a following important property of Leibniz kernel:

$$[[a, a], x] = [a, [a, x]] - [a, [a, x]] = 0.$$

This property shows that $\mathbf{Leib}(L)$ is an abelian subalgebra of L.

As usual, we say that a Leibniz algebra L is *finite dimensional*, if the dimension L as a vector space over F is finite.

The first step in the study of all types of algebras is the description of such algebras having small dimensions.

If $\mathbf{dim}_F(L) = 1$, then L is an abelian Lie algebra, that is $L = Fa$ for some element a and $[a, a] = 0$.

If $\mathbf{dim}_F(L) = 2$ and L is not a Lie algebra, then there are the following two non-isomorphic Leibniz algebras:

$$L_1 = Fa + Fb, [a, a] = b, [b, a] = [a, b] = [b, b] = 0,$$

and

$$L_2 = Fc + Fd, [c, c] = [c, d] = d, [d, c] = [d, d] = 0.$$

(see, for example, a survey [12]). The structure of 3-dimensional Leibniz algebras is more complicated. The study of Leibniz algebras, having dimension 3, has been conducted in the papers ([1,2,8,10]) for the fields of characteristic 0, moreover for a field $\mathbb{C}$ of complex numbers or algebraically closed field of characteristic 0. In this paper, we consider the opposite situation, where the structure of Leibniz algebras of dimension 3 over the finite field will be described. As we will see later, the situation here is much more diverse, in some cases the structure of the algebra essentially depends on the characteristic of the field, in others on the solvability of specific equations in the field, and so on. We will see that the Leibniz algebras of dimension 3 is soluble, therefore a first natural step of our study is a consideration of nilpotent algebras.

Let L be a Leibniz algebra. Define the *lower central series*

$$L = \gamma_1(L) \geqslant \gamma_2(L) \geqslant \ldots \gamma_\alpha(L) \geqslant \gamma_{\alpha+1}(L) \geqslant \ldots \gamma_\delta(L)$$

of L by the following rule: $\gamma_1(L) = L$, $\gamma_2(L) = [L, L]$, and recursively $\gamma_{\alpha+1}(L) = [L, \gamma_\alpha(L)]$ for all ordinals α and $\gamma_\lambda(L) = \bigcap\limits_{\mu<\lambda} \gamma_\mu(L)$ for the limit ordinals λ.It is possible to shows that every term of this series is an ideal of L. The last term $\gamma_\delta(L)$ is called the *lower hypocenter* of L. We have $\gamma_\delta(L) = [L, \gamma_\delta(L)]$.

If $\alpha = k$ is a positive integer, then $\gamma_k(L) = [L, [L, [L, \ldots] \ldots]]$.

A Leibniz algebra L is called *nilpotent* if there exists a positive integer k such that $\gamma_k(L) = \langle 0 \rangle$. More precisely, L is said to be *nilpotent of nilpotency class* $\mathbf{c}$ if $\gamma_{\mathbf{c}+1}(L) = \langle 0 \rangle$, but $\gamma_{\mathbf{c}}(L) \neq \langle 0 \rangle$. We denote by $\mathbf{ncl}(L)$ the nilpotency class of L.

The *left* (respectively *right*) *center* $\zeta^{left}(L)$ (respectively $\zeta^{right}(L)$) of a Leibniz algebra L is defined by the rule:

$$\zeta^{left}(L) = \{x \in L | \ [x, y] = 0 \text{ for each element } y \in L\}$$

(respectively,

$$\zeta^{right}(L) = \{x \in L|\ [y,x] = 0 \text{ for each element } y \in L\}).$$

It is not hard to prove that the left center of L is an ideal, but it is not true for the right center. Moreover, $\mathbf{Leib}(L) \leqslant \zeta^{left}(L)$, so that $L/\zeta^{left}(L)$ is a Lie algebra. The right center is a subalgebra of L, and in general, the left and right centers are different; they even may have different dimensions. The corecponding examples could be find in [13].

Of course we will consider a case when L is a not Lie algebra.

1. Nilpotent Leibniz algebra of dimension 3

In this section we will suppose that L is nilpotent.

Since $\mathbf{ncl}(L) \leqslant \mathbf{dim}_F(L)$, $\mathbf{ncl}(L) \leqslant 3$.

Let L be a Leibniz algebra over field F. The intersection of the maximal subalgebras of L is called the *Frattini subalgebra* of L and denoted by $\mathbf{Frat}(L)$. If L does not include maximal subalgebras, then put $L = \mathbf{Frat}(L)$.

We will need the following important property of Frattini subalgebras.

Proposition 1. *Let L be a finite dimensional Leibniz algebra over finite field F. If L is nilpotent then $[L,L] = \mathbf{Frat}(L)$.*

Indeed since L is nilpotent, every maximal subalgebra of L is an ideal [3, Lemma 2.2], so we can apply Proposition 7 of paper [12].

Theorem 1. *Let L be a nilpotent Leibniz algebra over finite field F. If L is a not Lie algebra and $\mathbf{ncl}(L) = 3 = \mathbf{dim}_F(L)$, then L has a basis $\{a,b,c\}$ such that $[a,a] = b$, $[a,b] = c$, $[c,a] = [a,c] = [b,a] = [c,b] = [b,c] = [b,b] = [c,c] = 0$. Moreover, $\mathbf{Leib}(L) = \zeta^{left}(L) = [L,L] = Fb \oplus Fc$, $\zeta^{right}(L) = \zeta(L) = \gamma_3(L) = Fc$. In particular, L is a nilpotent cyclic Leibniz algebra.*

Proof. Since $\gamma_1(L) = L \neq \gamma_2(L) = [L,L] \neq \gamma_3(L) \neq \langle 0\rangle$, $\mathbf{dim}_F(\gamma_2(L)) = 2$, $\mathbf{dim}_F(\gamma_3(L)) = 1 = \mathbf{dim}_F(L/\gamma_2(L))$. Let a be an element of L such that $a \notin \gamma_2(L)$. Then Proposition 1 shows that L is a cyclic algebra, generated by an element a. Put $b = [a,a]$, then $b \in \gamma_2(L) = [L,L]$ and $b \in \mathbf{Leib}(L)$. It follows that $[b,a] = 0$. If we suppose that $[a,b] = 0$, then $b \in \zeta(L)$. But in this case $\gamma_2(L) \leqslant \zeta(L)$ and $\gamma_3(L) = \langle 0\rangle$, that contradicts to our assumption. Thus $[a,b] = c \neq 0$. Then $c \in \gamma_3(L)$, that follows that $\gamma_3(L) = Fc$ and $[c,a] = [a,c] = [c,b] = [b,c] = 0$.

The case when $\mathbf{ncl}(L) = 2$ break up on two subcases. First: there exists an element $d \notin \gamma_2(L) = [L, L]$ such that $[d, d] = 0$. Second: $[d, d] \neq 0$ for each element $d \notin \gamma_2(L)$. In the second case $[d, d]$ is a non-zero element of $\zeta(L)$. Then $\zeta(L) = F[d, d] \leqslant \langle d \rangle$. Since a factor-algebra $L/\zeta(L)$ is abelian, a cyclic subalgebra $\langle d \rangle$ is an ideal. Then and every non-zero subalgebra of L is an ideal. □

The following two theorems consider the both these subcase separately.

Further writing $L = A \oplus B$ means that L is a direct sum of the subspaces A and B or the subalgebras A and B. If $L = A \oplus B$ and A is an ideal of L and B is a subalgebra of L, then we will say that L is a semidirect sum of A and B and use the following symbol $L = A \dashv B$.

Theorem 2. *Let L be a nilpotent Leibniz algebra over finite field F. Suppose that L is a not Lie algebra, $\mathbf{dim}_F(L) = 3$, $\mathbf{ncl}(L) = 2$ and L has an element $b \notin \gamma_2(L)$ such that $[b, b] = 0$. Then L is an algebra of one of the following types:*

I. *$L = A \oplus B$, where A, B are the ideals, $B = Fb$, $[b, b] = 0$, $A = Fa \oplus Fc$ is a cyclic nilpotent subalgebra, $[a, a] = c$, $[c, a] = [a, c] = [c, c] = 0$. Moreover, $\mathbf{Leib}(L) = [L, L] = Fc$, $\zeta^{left}(L) = \zeta^{right}(L) = \zeta(L) = Fb \oplus Fc$.*

II. *$L = A \dashv B$, where $A = Fa \oplus Fc$ is a cyclic nilpotent subalgebra, $[a, a] = c$, $[c, a] = [a, c] = 0$, B is an abelian subalgebra, $B = Fb$, $[b, b] = 0$, and $[a, b] = c$, $[b, a] = [b, c] = [c, b] = [c, c] = 0$. Moreover, $\mathbf{Leib}(L) = [L, L] = \zeta^{right}(L) = \zeta(L) = Fc$, $\zeta^{left}(L) = Fb \oplus Fc$.*

III. *$L = A \dashv B$, where $A = Fa \oplus Fc$ is a cyclic nilpotent subalgebra, $[a, a] = c$, $[c, a] = [a, c] = 0$, B is an abelian subalgebra, $B = Fb$, $[b, b] = 0$, and $[a, b] = c$, $[b, a] = \gamma c$, $\gamma \neq 0$, $[b, c] = [c, b] = [c, c] = 0$. Moreover, $\mathbf{Leib}(L) = [L, L] = \zeta^{left}(L) = \zeta^{right}(L) = \zeta(L) = Fc$.*

Proof. Since $\gamma_3(L) = \langle 0 \rangle$, then $\gamma_2(L) = [L, L] \leqslant \zeta(L)$. An inclusion $\mathbf{Leib}(L) \leqslant \gamma_2(L)$ shows that $\mathbf{dim}_F(\gamma_2(L)) \geqslant 1$. If we assume that $\mathbf{dim}_F(L/\gamma_2(L)) = 1$, then Proposition 1 shows that L is a cyclic algebra. But in this case it is not hard to prove that $\mathbf{dim}_F(L) = 2$. Hence $L/\gamma_2(L)$ is not cyclic, so that $\mathbf{dim}_F(L/\gamma_2(L)) = 2$. Put $K = \gamma_2(L)$. Since L is not a Lie algebra, there is an element a such that $[a, a] = c \neq 0$. An inclusion $K \leqslant \zeta(L)$ shows that $a \notin K$. Since $\mathbf{dim}_F(L/\gamma_2(L)) = 2$, $K = Fc$. The fact that L/K is abelian implies that $[a, x]$, $[x, a] \in K = Fc \leqslant \langle a \rangle$ for each element $x \in L$. This shows that a subalgebra $A = \langle a \rangle = Fa \oplus Fc$ is an ideal of L.

By the choice of an element a we obtain that $b \notin \langle a \rangle$, so that $L = A \oplus Fb$. An equality $[b, b] = 0$ shows that a subalgebra $\langle b \rangle$ is abelian, so that $\langle b \rangle = Fb$. Thus L is a semidirect sum of an ideal $\langle a \rangle$ and 1-dimensional abelian subalgebra Fb. Here we will have the following possibilities for commutators $[a, b]$ and $[b, a]$.

(i) $[a, b] = [b, a] = 0$, in this case a subalgebra $\langle b \rangle$ is an ideal and L is a direct sum of two ideals: $L = (Fa \oplus Fc) \oplus Fb$, so that L is a Leibniz algebra of type I.

(ii) $[a, b] = \alpha c$, $[b, a] = \beta c$, where $(\alpha, \beta) \neq (0, 0)$. If $\alpha \neq 0$ (respectively $\beta \neq 0$), then we can replace an element b on $b_1 = \alpha^{-1} b$ (respectively on $b_1 = \beta^{-1} b$). Clearly $L = \langle a \rangle \oplus Fb_1$ and $[a, b_1] = c$, $[b_1, b_1] = 0$ (respectively $[b_1, a] = c$, $[b_1, b_1] = 0$). Consider now two situations, which appear here:
(iia) $[a, b] = c, [b, a] = 0$,
(iib) $[a, b] = 0, [b, a] = c$.

In the second case put $a_1 = a$, $b_1 = a - b$, then we have:

$$[b_1, b_1] = [a - b, a - b] = [a, a] - [b, a] - [a, b] + [b, b] = c - c = 0,$$
$$[a_1, b_1] = [a, a - b] = [a, a] - [a, b] = c - 0 = c,$$
$$[b_1, a_1] = [a - b, a] = [a, a] - [b, a] = c - c = 0.$$

This shows that in every of situation *(iia)* and *(iib)* we come to the same algebra. Thus we obtain the Leibniz algebra of types II and III. □

In the second subcase we consider a situation when $[d, d]$ is a non-zero element of $\zeta(L)$ for each element $d \notin \zeta(L)$. As we noted above, in this case every non-zero subalgebra of L is an ideal.

Theorem 3. *Let L be a nilpotent Leibniz algebra over finite field F. Suppose that L is a not Lie algebra, $\mathbf{dim}_F(L) = 3$, $\mathbf{ncl}(L) = 2$ and $[d, d] \neq 0$ for each element $d \notin \gamma_2(L)$. Then L is an algebra of one of the following types:*

I. *$L = A + B$, where A, B are the nilpotent ideals, $A = \langle a \rangle$, $B = \langle b \rangle$, $A \cap B = \zeta(L) = Fc$, $[a, a] = [b, b] = c$, $[c, a] = [a, c] = [c, b] = [b, c] = [a, b] = [b, a] = [c, c] = 0$. Moreover, $\mathbf{Leib}(L) = [L, L] = \zeta^{left}(L) = \zeta^{right}(L) = \zeta(L) = Fc$, $\mathbf{char}(F) \neq 2$, and an equation $X^2 + 1 = 0$ has not a solution in F.*

II. *$L = A + B$, where A, B are the nilpotent ideals, $A = \langle a \rangle$, $B = \langle b \rangle$, $A \cap B = \zeta(L) = Fc$, $[a, a] = c$, $[b, b] = \rho c$, where ρ is a primitive root of identity of degree $|F| - 1$, $[c, a] = [a, c] = [c, b] = [b, c] =$*

$[a,b]=[b,a]=[c,c]=0$. *Moreover,* $\mathbf{Leib}(L)=[L,L]=\zeta^{left}(L)=\zeta^{right}(L)=\zeta(L)=Fc$, $\mathbf{char}(F)\neq 2$.

III. $L=A+B$, *where* A, B *are the nilpotent ideals,* $A=\langle a\rangle$, $B=\langle b\rangle$, $A\cap B=\zeta(L)=Fc$, $[a,a]=c=[a,b]$, $[b,b]=\eta c$, $[c,a]=[a,c]=[c,b]=[b,c]=[b,a]=[c,c]=0$. *Moreover,* $\mathbf{Leib}(L)=[L,L]=\zeta^{left}(L)=\zeta^{right}(L)=\zeta(L)=Fc$ *and a polynomial* $X^2+X+\eta$ *has no roots in a field* F.

Proof. As in Theorem 2 $\gamma_2(L)=[L,L]\leqslant\zeta(L)$, $\mathbf{Leib}(L)\leqslant\gamma_2(L)$, $\mathbf{dim}_F(\gamma_2(L))=1$, so that $\mathbf{dim}_F(L/\gamma_2(L))=2$. Put $K=\gamma_2(L)$. Let $a\notin K$, then $[a,a]=c\neq 0$. Since $\mathbf{dim}_F(\gamma_2(L))=1$, $K=Fc$. The fact that L/K is abelian implies that $[a,x]$, $[x,a]\in K=Fc\leqslant\langle a\rangle$ for each element $x\in L$. This shows that a subalgebra $A=\langle a\rangle=Fa\oplus Fc$ is an ideal of L.

Choose an element $b\notin\langle a\rangle$. Then the subset $\{c,a,b\}$ is a basis of L. By our condition $[b,b]=\eta c$ and $\eta\neq 0$. Consider the elements $[a,b]$ and $[b,a]$. Suppose first that $[a,b]=[b,a]=0$. Since F is finite, $G=F\backslash\{0\}$ is a cyclic group by multiplication of order $q=|F|-1$. Suppose that $\eta\in G^2$. Then $\eta=\kappa^2$ for some element $0\neq\kappa\in F$. Put $b_1=\kappa^{-1}b$, then

$$[b_1,b_1]=[\kappa^{-1}b,\kappa^{-1}b]=\kappa^{-2}[b,b]=\kappa^{-2}\eta c=\kappa^{-2}\kappa^2c=c.$$

In particular, if $\mathbf{char}(F)=2$, then $G^2=G$. We take an arbitrary element $d=\lambda a+\mu b_1$, where $\lambda\neq 0$, $\mu\neq 0$ and find $[d,d]$. By such choice $d\notin\zeta(L)$, so that $[d,d]\neq 0$. We have

$$\begin{aligned}[d,d]&=[\lambda a+\mu b_1,\lambda a+\mu b_1]\\&=\lambda^2[a,a]+\lambda\mu[b_1,a]+\lambda\mu[a,b_1]+\mu^2[b_1,b_1]\\&=\lambda^2c+\mu^2c.\end{aligned}$$

Suppose that $\lambda^2+\mu^2=0$. Since $\mu\neq 0$, we obtain $y^2+1=0$ where $y=\lambda\mu^{-1}$. Thus we see that a polynomial X^2+1 has no roots in a field F. Since in a field F such that $\mathbf{char}(F)=2$ a polynomial X^2+1 has root 1, the characteristic of F must be not 2. Thus we obtain the Leibniz algebras of type I.

For example, if $F=\mathbb{F}_5$, $\mathbb{F}_{13}$, an equation $y^2+1=0$ has a solution, if $F=\mathbb{F}_3$, $\mathbb{F}_7$, $\mathbb{F}_{11}$, an equation $y^2+1=0$ has not a solution. Thus we can see that the algebras have the same defining relation, but have the different properties.

Consider the case when $\eta\notin G^2$. Since G is cyclic, then G/G^2 is a cyclic group of order 2, so that $G=G^2\cup\rho G^2$, where ρ is a primitive root of identity of degree q. Then $\eta = \rho\kappa^2$ for some element $\kappa \in F$. Put $b_1 = \kappa^{-1}b$, then

$$[b_1, b_1] = [\kappa^{-1}b, \kappa^{-1}b] = \kappa^{-2}[b, b] = \kappa^{-2}\eta c = \kappa^{-2}\rho\kappa^2 c = \rho c.$$

As we noted above in this case $\mathbf{char}(F) \neq 2$. Let again $d = \lambda a + \mu b_1$, where $\lambda \neq 0, \mu \neq 0$. We have

$$\begin{aligned}[d, d] &= [\lambda a + \mu b_1, \lambda a + \mu b_1] = \lambda^2[a, a] + \mu^2[b_1, b_1] \\ &= \lambda^2 c + \mu^2\rho c = (\lambda^2 + \mu^2\rho)c.\end{aligned}$$

Suppose that $\lambda^2 + \mu^2\rho = 0$. Since $\mu \neq 0$, we obtain $y^2 + \rho = 0$, where $y = \lambda\mu^{-1}$. By the choice ρ is a generator of a multiplicative cyclic group $G = F\backslash\{0\}$, so that $\rho \notin G^2$. Since $\langle\rho\rangle \leqslant \langle-\rho\rangle$, $-\rho \notin G^2$. It follows that an equation $y^2 + \rho = 0$ has not a solution in F. Then $[d, d] \neq 0$ for every element $d \notin Fc$. As we have seen above in this case every subalgebra of L is an ideal. Thus we come to a Leibniz algebra of type II.

Suppose that $[b, a] = \alpha c \neq 0$. Then $\alpha \neq 0$. Put $b_1 = \alpha a - b$, then $[b_1, a] = [\alpha a - b, a] = \alpha[a, a] - [b, a] = \alpha c - \alpha c = 0$. Therefore further we will suppose that $[b, a] = 0$.

We have $[a, b] = \beta c$ for some element $\beta \in F$. The case when $\beta = 0$ we have already considered. Assume therefore that $[a, b] = \beta c \neq 0$. Put $b_1 = \beta^{-1}b$, then $[a, b_1] = [a, \beta^{-1}b] = \beta^{-1}[a, b] = \beta^{-1}\beta c = c$. Let again $d = \lambda a + \mu b$ where $\lambda \neq 0, \mu \neq 0$. We have

$$\begin{aligned}[d, d] &= [\lambda a + \mu b_1, \lambda a + \mu b_1] == \lambda^2[a, a] + \lambda\mu[a, b_1] + \mu^2[b_1, b_1] \\ &= \lambda^2 c + \lambda\mu c + \mu^2\eta c = (\lambda^2 + \lambda\mu + \mu^2\eta)c, \quad \text{where } \eta = [b_1, b_1].\end{aligned}$$

Suppose that $\lambda^2 + \lambda\mu + \mu^2\eta = 0$. Since $\mu \neq 0$, we obtain an equation $y^2 + y + \eta = 0$ where $y = \lambda\mu^{-1}$. Hence we see that a polynomial $X^2 + X + \eta$ has no roots in a field F. Thus we obtain the Leibniz algebra of type III.

As we can see, in this last case the properties of the algebra depend on whether the polynomial above has roots in the field F, and this also depends on the choice of the element η. The difference appears even over fields of the same characteristic. So, if $F = \mathbb{F}_2$, then for η only one value is possible $\eta = 1$. But an equation $y^2 + y + 1 = 0$ has no solution in a field $F = \mathbb{F}_2$. Hence the respectively algebra has no element $d \notin Fc$ such that $[d, d] = 0$. It follows that every subalgebra of L is an ideal. If $F = \mathbb{F}_4$ and $\eta = 1$, then an equation $y^2 + y + 1 = 0$ has solutions in a field $\mathbb{F}_4$. In this case the respectively algebra L has one dimensional subalgebra, which is no ideal. $\square$

2. Non-nilpotent Leibniz algebras of dimension 3 with 1-dimensional Leibniz kernel over finite fields

The next step is a consideration of a case when L is non-nilpotent. We will consider a Leibniz algebras of dimension 3 over finite fields, which are not Lie algebras. It follows that $\mathbf{Leib}(L) \neq \langle 0 \rangle$. Since $\mathbf{Leib}(L)$ is an abelian ideal, $L \neq \mathbf{Leib}(L)$. Hence for $\mathbf{Leib}(L)$ we have only two possibility: $\mathbf{dim}_F(\mathbf{Leib}(L)) = 1$, $\mathbf{dim}_F(\mathbf{Leib}(L)) = 2$.

In this section we consider the case when $\mathbf{dim}_F(\mathbf{Leib}(L)) = 1$, so that $\mathbf{dim}_F(L/\mathbf{Leib}(L)) = 2$.

Theorem 4. *Let L be a non-nilpotent Leibniz algebra over finite field F. Suppose that L is a not Lie algebra, $\mathbf{dim}_F(L) = 3$ and $\mathbf{dim}_F(\mathbf{Leib}(L)) = 1$. Then L is an algebra of one of the following types:*

I. *$L = A \oplus B$, where A, B are the ideals, $B = Fb$, $[b,b] = 0$, A is a cyclic subalgebra, $A = Fa \oplus Fc$, where $[a,a] = c = [a,c]$, $[c,a] = [c,b] = [b,c] = [a,b] = [b,a] = [c,c] = 0$. Moreover, $\mathbf{Leib}(L) = [L,L] = Fc$, $\zeta^{left}(L) = Fb \oplus Fc$, $\zeta^{right}(L) = \zeta(L) = Fb$.*

II. *$L = A \dashv B$, where $B = Fb$, $[b,b] = 0$, $A = Fa \oplus Fc$ is a cyclic subalgebra, $[a,a] = c = [a,c]$, $[a,b] = c$, $[c,a] = [c,b] = [b,c] = [b,a] = [c,c] = 0$. Moreover, $\mathbf{Leib}(L) = [L,L] = Fc$, $\zeta^{left}(L) = Fb \oplus Fc$, $\zeta(L) = \zeta^{right}(L) = \langle 0 \rangle$.*

III. *$L = A \dashv B$, where $B = Fb$, $[b,b] = 0$, $A = Fa \oplus Fc$ is a cyclic subalgebra, $[a,a] = c = [a,c]$, $[b,a] = [b,c] = c$, $[c,a] = [c,b] = [a,b] = [c,c] = 0$. Moreover, $\mathbf{Leib}(L) = [L,L] = \zeta^{left}(L) = Fc$, $\zeta^{right}(L) = Fb$, $\zeta(L) = \langle 0 \rangle$.*

IV. *$L = A \dashv B$,, where $B = Fb$, $[b,b] = 0$, $A = Fa \oplus Fc$ is a cyclic subalgebra, $[a,a] = c = [a,c]$, $[a,b] = a = -[b,a]$, $[b,c] = -2c$, $[c,a] = [c,b] = [c,c] = 0$. Moreover, $\mathbf{Leib}(L) = [L,L] = \zeta^{left}(L) = Fc$, $\zeta^{right}(L) = \zeta(L) = \langle 0 \rangle$.*

V. *$L = A \dashv B$, where $B = Fb$, $[b,b] = 0$, $A = Fa \oplus Fc$ is a cyclic subalgebra, $[a,a] = c$, $[a,c] = 0$, $[a,b] = a+\gamma c$, $\gamma \in F$, $[b,a] = -a+\gamma c$, $[b,c] = -2c$, $[c,a] = [c,b] = 0$. Moreover, $\mathbf{Leib}(L) = [L,L] = \zeta^{left}(L) = Fc$, $\zeta^{right}(L) = \zeta(L) = \langle 0 \rangle$ whenever $\mathbf{char}(F) \neq 2$ and $\zeta^{right}(L) = \zeta(L) = Fc$ whenever $\mathbf{char}(F) = 2$.*

Proof. Since L is not Lie algebra, there is an element a such that $[a,a] = c \neq 0$. Then $c \in K = \mathbf{Leib}(L)$, so that $K = Fc$. An inclusion $\mathbf{Leib}(L) \leqslant \zeta^{left}(L)$ implies that $[c,y] = 0$ for each element $y \in L$, in particular, $[c,a] = 0$. A subalgebra $\langle a \rangle$ has dimension 2, therefore either $[a,c] = 0$ or $[a,c] = c$ (see, for example, a survey [12]). Suppose that a Lie algebra L/K

is abelian and let first $[a,c]=0$. Also assume that there is an element $b \notin \langle a\rangle$ such that $[b,b]=0$. The fact that L/K is abelian implies that $[a,b]$, $[b,a] \in Fc$. If $[a,b]=0=[b,a]$, then a subalgebra $\langle b\rangle = Fb$ is an ideal and L is a direct sum of two subalgebras: $L=(Fa\oplus Fc)\oplus Fb$. Since L is non-nilpotent, then for $[a,c]$ only one variant remains: $[a,c]=c$. Thus we obtain the Leibniz algebra of type I.

Let now $[a,b]=\alpha c$, $[b,a]=\beta c$, where $(\alpha,\beta)\neq(0,0)$. If $\alpha\neq 0$ (respectively $\beta\neq 0$), then we can replace an element b on $b_1=\alpha^{-1}b$ (respectively on $b_1=\beta^{-1}b$). Clearly $L=\langle a\rangle\oplus Fb_1$ and $[a,b_1]=c$, $[b_1,b_1]=0$ (respectively $[b_1,a]=c$, $[b_1,b_1]=0$). Consider further all three situation, which appear here.

(a) Suppose that $[a,b]=c$, $[b,a]=0$. Then

$$[b,c]=[b,[a,a]]=[[b,a],a]+[a,[b,a]]=0.$$

If we assume that $[a,c]=0$, then $K\leqslant\zeta(L)$, which follows that an algebra L is nilpotent. This situation was considered early. Thus we obtain the Leibniz algebra of type II.

(b) Suppose that $[a,b]=0$, $[b,a]=c$. Then

$$[b,c]=[b,[a,a]]=[[b,a],a]+[a,[b,a]]=[c,a]+[a,c]=[a,c].$$

If we assume that $[a,c]=0$, then $K\leqslant\zeta(L)$, so we again come to nilpotent case. Thus we obtain the Leibniz algebra of type III.

(c) Suppose that $[a,b]=c$, $[b,a]=\beta c$, where $\beta\neq 0$. Then

$$[b,c]=[b,[a,a]]=[[b,a],a]+[a,[b,a]]=\beta[c,a]+\beta[a,c]=\beta[a,c].$$

On the other hand,

$$[b,c]=[b,[a,b]]=[[b,a],b]+[a,[b,b]]=\beta[c,b]+[a,0]=0.$$

If we assume that $[a,c]=0$, then $K\leqslant\zeta(L)$, so we again come to nilpotent case. Then an equality $[b,c]=\beta[a,c]$ implies that $\beta=0$. Thus we obtain a contradiction, which shows that the last case is impossible.

Suppose now that $[d,d]\neq 0$ for each element $d\notin K$. It follows that $[d,d]$ is a non-zero element of K. Then $K=F[d,d]\leqslant\langle d\rangle$. Since a factor-algebra L/K is abelian, a cyclic subalgebra $\langle d\rangle$ is an ideal. If $0\neq d\in K$, then $\langle d\rangle = K$, in particular, $\langle d\rangle$ again is an ideal. Then and every non-zero subalgebra of L is an ideal. But in this case an algebra L must be nilpotent ([14, Theorem A]). This case was considered above.

Assume now that $\mathbf{dim}_F(\mathbf{Leib}(L)) = 1$ and a Lie algebra L/K is non-abelian. Again there exists an element a such that $[a, a] \neq 0$. Since L/K is a Lie algebra, $[a, a] \in K$. If we suppose that $a + K \in \zeta(L/K)$, then an equality $\mathbf{dim}_F(L/K) = 2$ implies that L/K is abelian, and we obtain a contradiction. Thus we can find the element y such that $K \neq [a + K, y + K]$. The fact that L/K is a Lie algebra of dimension 2 implies that $L/K \neq [L/K, L/K]$ (see, for example, [11, Chapter 1, Section 4]). Then $\mathbf{dim}_F([L/K, L/K]) = 1$. Put $d = [a, y]$, then $[L/K, L/K] = \langle d + K\rangle = D/K$. As above $d + K \notin \zeta(L/K)$. We have one of two following possibilities: $\langle d + K\rangle = \langle a + K\rangle$ or the intersection $\langle d + K\rangle \cap \langle a + K\rangle$ is zero.

Consider a first possibility, let $\langle d + K\rangle = \langle a + K\rangle$. Suppose that there exists an element $b \notin D$ such that $[b, b] = 0$. Then $L = D \oplus \langle b\rangle$, where a subalgebra $\langle b\rangle$ has a dimension one, that is $\langle b\rangle = Fb$. Since $c = [a, a] \in K$, $D = Fa \oplus Fc$. Since $a + K \notin \zeta(L/K)$, $[a + K, b + K] \neq K$, that is $[a+K, b+K] = \delta(a+K)$ for some non-zero element $\delta \in F$. Put $b_1 = \delta^{-1}b$, then

$$\begin{aligned}[a + K, b_1 + K] &= [a + K, \delta^{-1}b + K] = \delta^{-1}[a + K, b + K]\\ &= \delta^{-1}(d + K) = \delta^{-1}\delta(a + K) = a + K.\end{aligned}$$

Since $[b_1, b_1] = \delta^{-2}[b, b] = 0$, we can assume further that $[a + K, b + K] = a + K$. It follows that $[a, b] = a + \gamma c$ for some element $\gamma \in F$. Since L/K is a Lie algebra, $[b + K, a + K] = -[a + K, b + K] = -a + K$, so that $[b, a] = -a + \gamma_1 c$ for some element $\gamma_1 \in F$. An inclusion $c \in \mathbf{Leib}(L)$ implies that $[c, x] = 0$ for all $x \in L$, in particular, $[c, a] = [c, b] = 0$. For element $[b, c]$ we obtain:

$$\begin{aligned}[b, c] &= [b, [a, a]] = [[b, a], a] + [a, [b, a]] = [-a + \gamma_1 c, a] + [a, -a + \gamma_1 c]\\ &= [-a, a] + [a, -a] = -2c.\end{aligned}$$

Further,

$$\begin{aligned}[[a, b], b] &= [a, [b, b]] - [b, [a, b]] = -[b, [a, b]] = -[b, a + \gamma c]\\ &= -[b, a] - \gamma[b, c] = a - \gamma_1 c + 2\gamma c = a + (2\gamma - \gamma_1)c,\end{aligned}$$

and $[[a, b], b] = [a + \gamma c, b] = [a, b] = a + \gamma c$, so that $a + (2\gamma - \gamma_1)c = a + \gamma c$, which implies that $\gamma_1 = \gamma$.

A subalgebra $\langle a\rangle$ has dimension 2, therefore we have two cases:

(i) $[a, c] = 0$ and (ii) $[a, c] = c$ (see, for example, a survey [12]).

If $[a,c]=c$, then

$$\begin{aligned}[[b,a],a] &= [b,[a,a]]-[a,[b,a]] = [b,c]-[a,-a+\gamma_1 c]\\ &= -2c+[a,a]-\gamma_1[a,c] = -2c+c-\gamma_1[a,c],\end{aligned}$$

and $[[b,a],a]=[-a+\gamma_1 c,a]=-c$, which implies that $-2c+c-\gamma_1[a,c]=-c$, so that $\gamma_1=0$. Hence if $[a,c]=c$, then we obtain the Leibniz algebra of type IV.

If $[a,c]=0$, then we obtain the Leibniz algebra of type V.

Assume now that $[b,b]\neq 0$ for each element $b\notin D$. As above we can choose an element $b\notin D$ such that $[a+K,b+K]=a+K$. It follows that $[a,b]=a+\gamma c$ for some element $\gamma\in F$. Since L/K is a Lie algebra,

$$[b+K,a+K]=-[a+K,b+K]=-a+K,$$

so that $[b,a]=-a+\gamma_1 c$ for some element $\gamma_1\in F$. Since $[b,b]\neq 0$, $[b,b]=\eta c$ for some non-zero element $\eta\in F$. An inclusion $c\in \mathbf{Leib}(L)$ implies that $[c,x]=0$ for all $x\in L$, in particular, $[c,a]=[c,b]=0$. For element $[b,c]$ we obtain again

$$\begin{aligned}[b,c] &= [b,[a,a]]=[[b,a],a]+[a,[b,a]]=[-a+\gamma_1 c,a]+[a,-a+\gamma_1 c]\\ &= [-a,a]+[a,-a]=-2c.\end{aligned}$$

Further,

$$\begin{aligned}[[a,b],b] &= [a,[b,b]]-[b,[a,b]]=[a,\eta c]-[b,[a,b]]=\eta[a,c]-[b,a+\gamma c]\\ &= \eta[a,c]-[b,a]-\gamma[b,c]=\eta[a,c]+a-\gamma_1 c+2\gamma c\\ &= \eta[a,c]+a+(2\gamma-\gamma_1)c.\end{aligned}$$

If $[a,c]=0$, then $[[a,b],b]=a+(2\gamma-\gamma_1)c$, if $[a,c]=c$, then $[[a,b],b]=a+(\eta+2\gamma-\gamma_1)c$. On the other hand, $[[a,b],b]=[a+\gamma c,b]=[a,b]=a+\gamma c$, so that $a+(2\gamma-\gamma_1)c=a+\gamma c$ or $a+(\eta+2\gamma-\gamma_1)c=a+\gamma c$. Thus if $[a,c]=0$, then $\gamma_1=\gamma$, if $[a,c]=c$, then $\gamma_1=\eta+\gamma$.

Furthermore,

$$\begin{aligned}[[b,a],a] &= [b,[a,a]]-[a,[b,a]]=[b,c]-[a,-a+\gamma_1 c]\\ &= -2c+[a,a]-\gamma_1[a,c]=-2c+c-\gamma_1[a,c]=-c-\gamma_1[a,c].\end{aligned}$$

and $[[b,a],a]=[-a+\gamma_1 c,a]=-c$, which implies that $\gamma_1[a,c]=0$. In particular, if $[a,c]=c$ then $\gamma_1=0$. As we have seen above, in this case

$\gamma_1 = \eta + \gamma$, therefore $\eta = -\gamma$. Suppose that $[a, c] = c$ and consider an element $a + b + c$. We have

$$\begin{aligned}[a + b + c, a + b + c] &= [a, a] + [a, b] + [a, c] + [b, a] + [b, b] + [b, c] \\ &= c + a + \gamma c + c - a - \gamma c - 2c = 0.\end{aligned}$$

Since $a+b+c \notin D$, we obtain a contradiction, which shows that a variant when $[a, c] = c$ is impossible.

Hence must be $[a, c] = 0$. Let $x = \lambda a + \mu b + \nu c$ be an arbitrary element of L such that $x \notin D$, where λ, μ, ν are some elements of F. We have

$$\begin{aligned}&[\lambda a + \mu b + \nu c, \lambda a + \mu b + \nu c] \\ &\quad = \lambda^2[a, a] + \lambda\mu[a, b] + \lambda\mu[b, a] + \mu^2[b, b] + \mu\nu[b, c] \\ &\quad = \lambda^2 c + \lambda\mu(a + \gamma c) + \lambda\mu(-a + \gamma c) + \mu^2\eta c - 2\mu\nu c \\ &\quad = (\lambda^2 + 2\lambda\mu\gamma + \mu^2\eta - 2\mu\nu)c.\end{aligned}$$

Consider an equation $\lambda^2 + 2\lambda\mu\gamma + \mu^2\eta - 2\mu\nu = 0$. If F is a finite field of characteristic 2, then we obtain $\lambda^2 + \mu^2\eta = 0$. Put $\mu = 1$, by such choice $x \notin D$. An equation $\lambda^2 + \eta = 0$ has a solution in a field F, because $F = \mathbb{F}_2$. Thus if F is a finite field of characteristic 2, then there exists an element $x \notin D$ such that $[x, x] = 0$, and we obtain a contradiction with our assumption. Suppose now that $\mathbf{char}(F) \neq 2$. The fact that $\mathbf{char}(F) \neq 2$ implies that an equation $2x = a$ has a solution $\frac{1}{2}a$ for each element $a \in F$. Put $\lambda = \mu = 1$, so we come to equation $1 + 2\gamma + \eta - 2\nu = 0$. Thus if we put $\lambda = \mu = 1$, $\nu = \frac{1}{2}(1 + 2\gamma + \eta)$, then $x \notin D$ and $[x, x] = 0$, and we again obtain a contradiction. $\square$

3. Non-nilpotent cyclic Leibniz algebras of dimension 3 over finite fields

The next step is a consideration of a case when L is non-nilpotent or $\mathbf{dim}_F(\mathbf{Leib}(L)) = 2$. Here there appear two variants: L is a cyclic algebra and L is a non-cyclic algebra. In this section we will consider a case when Leibniz algebra of dimension 3 is cyclic.

Theorem 5. *Let L be a non-nilpotent cyclic Leibniz algebra of dimension 3 over finite field F. Then L is an algebra of one of the following types:*

I. *$L = D \dashv A$, where $D = Fd$, $[d, d] = 0$, $A = Fa \oplus Fc$ is a cyclic nilpotent subalgebra, $[a, a] = c$, $[a, c] = 0$, $[a, d] = \delta d$, $0 \neq \delta \in F$, $[c, a] = [c, d] = [c, c] = [d, c] = [d, a] = 0$. Moreover, $\mathbf{Leib}(L) = [L, L] = \zeta^{left}(L) = Fd \oplus Fc$, $\zeta(L) = \zeta^{right}(L) = Fc$.*

II. $L = D \dashv B$, *where* $B = Fb$, $[b, b] = 0$, $D = Fd \oplus Fc$ *is an abelian subalgebra,* $[d, d] = [d, c] = [c, d] = [c, c] = 0$, $[b, c] = d$, $[b, d] = \gamma d + \delta d$, $0 \neq \gamma,\ \delta \in F$, $[c, b] = [d, b] = 0$. *Moreover,* $\mathbf{Leib}(L) = [L, L] = \zeta^{left}(L) = Fd \oplus Fc$, $\zeta^{right}(L) = Fb$, $\zeta(L) = \langle 0\rangle$.

Proof. Since L is cyclic and has dimension 3, L can not be a Lie algebra. Let a be an element of L such that $L = \langle a\rangle$. Since L is a not Lie algebra, $[a, a] = c \neq 0$. Moreover, $c \in \mathbf{Leib}(L) \leqslant \zeta^{left}(L)$, that implies $[c, x] = 0$ for all $x \in L$, in particular, $[c, a] = 0$. Since $L/\mathbf{Leib}(L)$ is a cyclic Lie algebra, that $\mathbf{dim}_F(L/\mathbf{Leib}(L)) = 1$, so that $\mathbf{dim}_F(\mathbf{Leib}(L)) = 2$. If we assume that $[a, c] \in Fc$, then $\langle a\rangle = Fa \oplus Fc$, so that $\mathbf{dim}_F(L) = 2$, and we obtain a contradiction. Thus $d = [a, c] \notin Fc$, which follows that $\{c, d\}$ is a basis of $K = \mathbf{Leib}(L)$. Again $[d, a] = 0$. Since K is an abelian subalgebra, $[c, c] = [c, d] = [d, d] = [d, c] = 0$. Since K is an ideal, $[a, d] \in K$, so that $[a, d] = \gamma c + \delta d$. If $\gamma = 0 = \delta$, then L is a cyclic nilpotent algebra. This situation was considered early. Suppose now that $\gamma = 0$, $\delta \neq 0$, then $[a, d] \in Fd$, which follows that $D = Fd$ is an ideal of L. Put $a_1 = a - \delta^{-1}c$, then

$$
\begin{aligned}
[a_1, a_1] &= [a - \delta^{-1}c, a - \delta^{-1}c] = [a, a] - \delta^{-1}[a, c] = c - \delta^{-1}d = a_2,\\
[a_1, a_2] &= [a - \delta^{-1}1c, c - \delta^{-1}d] = [a, c] - \delta^{-1}[a, d] = d - \delta^{-1}\delta d = 0,\\
[a_1, d] &= [a - \delta^{-1}c, d] = [a, d] = \delta d.
\end{aligned}
$$

Thus we can see that a subalgebra $A = \langle a_1\rangle$ is nilpotent and has dimension 2, $D \cap A = \langle 0\rangle$, so that L is semidirect sum of an ideal D of dimension 1 and a nilpotent subalgebra A of dimension 2. Moreover, $D \leqslant \zeta^{left}(L)$, $[L, L] = \zeta^{left}(L) = \mathbf{Leib}(L) = D \oplus [A, A]$. Thus we obtain the Leibniz algebra of type I.

Suppose now that if $\gamma \neq 0, \delta \neq 0$. Thus $[a, d] = \gamma[a, a] + \delta[a, c]$ and $0 = [a, \gamma a + \delta c - d]$. Put $b = a + \gamma^{-1}\delta c - \gamma^{-1}d$, then $[a, b] = 0$. Further

$$[b, b] = [a + \gamma^{-1}\delta c - \gamma^{-1}d, b] = [a, b] + [\gamma^{-1}\delta c - \gamma^{-1}d, b] = 0,$$

so that $\langle b\rangle = Fb$. Since $[c, b] = [d, b] = 0$, $Fb = \zeta^{right}(L)$. Furthermore

$$
\begin{aligned}
[b, c] &= [a + \gamma^{-1}\delta c - \gamma^{-1}d, c] = [a, c] = d,\\
[b, d] &= [a + \gamma^{-1}\delta c - \gamma^{-1}d, d] = [a, d] = \gamma c + \delta d.
\end{aligned}
$$

We can see that L is semidirect sum of an abelian ideal $Fc \oplus Fd$ of dimension 2 and abelian subalgebra Fb of dimension 1. Moreover, $\zeta^{left}(L) = Fc \oplus Fd = \mathbf{Leib}(L) = [L, L]$, $\zeta^{right}(L) = Fb$. Thus we obtain the Leibniz algebra of type II. $\square$

4. Non-nilpotent Leibniz algebras of dimension 3 with 2-dimensional Leibniz kernel over finite fields

The last case of our consideration is the case when L is non-nilpotent, non-cyclic and $\mathbf{dim}_F(\mathbf{Leib}(L)) = 2$. Then $\mathbf{dim}_F(L/\mathbf{Leib}(L)) = 1$. In particular, $L/\mathbf{Leib}(L)$ is abelian.

Theorem 6. *Let L be a non-nilpotent non-cyclic Leibniz algebra of dimension 3 over finite field F. Suppose that L is a not Lie algebra and $\mathbf{dim}_F(\mathbf{Leib}(L)) = 2$. Then L is an algebra of one of the following types:*

I. *$L = A \dashv D$, where $D = Fd$, $[d,d] = 0$, $A = Fa \oplus Fc$ is a cyclic subalgebra, $[a,a] = c = [a,c]$, $[a,d] = d$, $[c,a] = [c,d] = [d,c] = [d,a] = [c,c] = 0$. Moreover, $\mathbf{Leib}(L) = [L,L] = \zeta^{left}(L) = Fd \oplus Fc$, $\zeta(L) = \zeta^{right}(L) = \langle 0 \rangle$.*

II. *$\mathbf{char}(F) \neq 2$, $L = A \dashv D$, where $D = Fd$, $[d,d] = 0$, $A = Fa \oplus Fc$ is a cyclic subalgebra, $[a,a] = c = [a,c]$, $[a,d] = c + 2d$, $[c,a] = [c,d] = [d,c] = [d,a] = 0$. Moreover, $\mathbf{Leib}(L) = [L,L] = \zeta^{left}(L) = Fd \oplus Fc$, $\zeta(L) = \zeta^{right}(L) = \langle 0 \rangle$.*

Proof. Put $K = \mathbf{Leib}(L)$. Since L is a not Lie algebra, there is an element a such that $[a,a] = c \neq 0$. Then $a \notin K$, so that $L = K \oplus Fa$. An inclusion $\mathbf{Leib}(L) \leqslant \zeta^{left}(L)$ implies that $[c,y] = 0$ for each element $y \in L$, in particular, $[c,a] = 0$. For an element $[a,c]$ we have the following possibilities: $[a,c] = 0$, $[a,c] \in Fc$, $[a,c] \notin Fc$.

Consider the first situation. Choose an element $d \in K$ such that $K = Fc \oplus Fd$. Since K is abelian, $[d,d] = 0 = [c,d] = [d,c]$. It follows that $c \in \zeta(L)$. We have $[d,a] = 0$ and $[a,d] = \gamma c + \delta d$. If $\gamma = 0$, then a subalgebra $\langle d \rangle = Fd$ is an ideal. In this case we obtain that $L = D \oplus A$, where $D = Fd$ is an abelian ideal, $D \leqslant \zeta^{left}(L)$, A is a nilpotent cyclic subalgebra, $A = Fa \oplus Fc$, where $[a,a] = c$, $[L,L] = D \oplus [A,A] = \zeta^{left}(L)$, is an abelian ideal. Thus we obtain the Leibniz algebra of above considered Type I from Theorem 5.

Assume now that $\gamma \neq 0$. Put $d_1 = \gamma^{-1} d$, then

$$[a,d_1] = \gamma^{-1}[a,d] = \gamma^{-1}(\gamma c + \delta d) = c + \delta d_1.$$

If we suppose that $\delta = 0$, then again $L/\zeta(L)$ is abelian and L is nilpotent. This case has been considered above. Therefore we assume that $\delta \neq 0$. We have

$$[a + d_1, a + d_1] = [a,a] + [a,d_1] = c + c + \delta d_1 = 2c + \delta d_1.$$

If $\mathbf{char}(F) = 2$, then $Fd_1 \leqslant \langle a + d_1 \rangle$, which follows that $a \in \langle a + d_1 \rangle$ and $c = [a, a] \in \langle a+d_1 \rangle$. Thus we can see, that in this case L is a cyclic algebra. Suppose that $\mathbf{char}(F) \neq 2$. Then from $\delta(a+d_1)-[a+d_1, a+d_1] \in \langle a+d_1 \rangle$, we obtain $\delta a-2c \in \langle a+d_1 \rangle$. It follows that $\delta^2 c = [\delta a-2c, \delta a-2c] \in \langle a+d_1 \rangle$, Since $\delta^2 \neq 0$, $Fc \leqslant \langle a + d_1 \rangle$. In turn out it follows that $Fd_1 \leqslant \langle a + d_1 \rangle$, and again we obtain that L is a cyclic.

Suppose now that $0 \neq [a, c] \in Fc$. Then a subalgebra $\langle a \rangle$ has dimension 2 and is non-nilpotent. Therefore we can suppose that $[a, c] = c$ (see, for example, a survey [12]). Choose an element $d \in K$ such that $K = Fc \oplus Fd$. Since K is abelian, $[d, d] = 0 = [c, d] = [d, c]$. We have $[d, a] = 0$ and $[a, d] = \gamma c + \delta d$. If $\gamma = 0$, then a subalgebra $\langle d \rangle = Fd$ is an ideal. In this case we obtain that $L = D \oplus A$, where $D = Fd$ is an abelian ideal, $D \leqslant \zeta^{left}(L)$, A is a non-nilpotent cyclic subalgebra of dimension 2, $A = Fa \oplus Fc$, where $[a, a] = c = [a, c]$, $[L, L] = D \oplus [A, A] = \zeta^{left}(L)$, is an abelian ideal. Thus we obtain the Leibniz algebra of type I.

Assume now that $\gamma \neq 0$. Put $d_1 = \gamma^{-1} d$, then

$$[a, d_1] = \gamma^{-1}[a, d] = \gamma^{-1}(\gamma c + \delta d) = c + \delta d_1.$$

If we suppose that $\delta = 0$, then $[a, d_1] \in Fc$, which implies that L/Fc is abelian. In turn out it follows that $\mathbf{Leib}(L) \leqslant Fc$. But in this case $\mathbf{dim}_F(\mathbf{Leib}(L)) = 1$, and we obtain a contradiction. This contradiction shows $\delta \neq 0$. We have

$$[a + d_1, a + d_1] = [a, a] + [a, d_1] = c + c + \delta d_1 = 2c + \delta d_1.$$

If $\mathbf{char}(F) = 2$, then $Fd_1 \leqslant \langle a + d_1 \rangle$, which follows that $a \in \langle a + d_1 \rangle$ and $c = [a, a] \in \langle a + d_1 \rangle$. Thus we can see, that in this case L is a cyclic algebra.

Suppose that $\mathbf{char}(F) \neq 2$. Then from $\delta(a + d_1) - [a + d_1, a + d_1] \in \langle a + d_1 \rangle$, we obtain $\delta a - 2c \in \langle a + d_1 \rangle$. It follows that

$$(\delta^2 - 2\delta)c = \delta^2 c - 2\delta c = [\delta a - 2c, \delta a - 2c] \in \langle a + d_1 \rangle.$$

By above $\delta \neq 0$. If $\delta \neq 2$, then $Fc \leqslant \langle a + d_1 \rangle$. In turn out it follows that $Fd_1 \leqslant \langle a + d_1 \rangle$, and again we obtain that L is a cyclic. Suppose that $\delta = 2$, that is $[a, d_1] = c + 2d_1$. Hence, if L is not cyclic, then L is a Leibniz algebra of a type II.

Suppose now that $[a, c] \notin Fc$. In this case $[a, c] = \alpha c + \beta d$ besides $\beta \neq 0$. We can see that a Leibniz algebra L is cyclic. This case has been considered above. $\square$

References

[1] S. Albeverio, B.A. Omirov, I.S. Rakhimov, *Varieties of nilpotent complex Leibniz algebras of dimension less than five*, Comm. Algebra **33** (2005), no. 5, 1575-1585.

[2] S.A. Ayupov, B.A. Omirov, *On 3-dimensional Leibniz algebras*, Uzbek. Math. Zh., **1** (1999), 9-14.

[3] D. Barnes, *Some theorems on Leibniz algebras*, Comm. Algebra **39** (2011), no. 7, 2463-2472.

[4] D. Barnes, *Schunck Classes of soluble Leibniz algebras*, Comm. Algebra **41** (2013), no. 11, 4046-4065.

[5] A.M. Bloh, *On a generalization of the concept of Lie algebra*, Dokl. Akad. Nauk SSSR **165** (1965), no. 3, 471-473.

[6] A.M. Bloh, *Cartan-Eilenberg homology theory for a generalized class of Lie algebras*, Dokl. Akad. Nauk SSSR **175** (1967), no. 8, 824-826.

[7] A.M. Bloh, *A certain generalization of the concept of Lie algebra*, Algebra and number theory. Moskov. Gos. Ped. Inst. Uchen. Zap. **375** (1971), 9-20.

[8] J.M. Casas, M.A. Insua, M. Ladra, S. Ladra, *An algorithm for the classification of 3-dimensional complex Leibniz algebras*, Linear Algebra Appl. **436** (2012), no. 9, 3747-3756.

[9] V.A. Chupordya, L.A. Kurdachenko, I.Ya. Subbotin, *On some "minimal" Leibniz algebras*, J. Algebra Appl. **16** (2017), no. 2.

[10] I. Demir, K.C. Misra, E. Stitzinger, *On some structures of Leibniz algebras*, Recent Advances in Representation Theory, Quantum Groups, Algebraic Geometry, and Related Topics, Contemporary Mathematics, **623** (2014), 41-54.

[11] N. Jacobson, *Lie algebras*, John Wiley: New York, (1962).

[12] V.V. Kirichinko, L.A. Kurdachenko, A.A. Pypka, I.Ya. Subbotin, *Some aspects of Leibniz algebra theory*, Algebra and Discrete Mathematics, V. 24, N.**1**, (2017), pp.1-33.

[13] L.A. Kurdachenko, J. Otal, A.A. Pypka, *Relationships between factors of canonical central series of Leibniz algebras*, Eur. J. Math. **2** (2016), no. 2, 565-577.

[14] L.A. Kurdachenko, N.N. Semko, I.Ya. Subbotin, *The Leibniz algebras whose subalgebras are ideals*, Open Math. **15** (2017), no. 1, 92-100.

[15] J.-L. Loday, *Une version non commutative des algèbres de Lie: les algèbras de Leibniz*, L'Enseignement Mathèmatique **39** (1993), 269-293.

[16] J.-L. Loday, *Cyclic homology.* Grundlehren der Mathematischen Wissenschaften, Vol. **301**, 2nd ed., Springer, Verlag, Berlin, 1998.

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