Definition 5.4. Let G be a finitely generated group, acting on a set A. *Growth degree* of the G-action is the number

$$\gamma = \sup_{w \in A} \limsup_{r \to \infty} \frac{\log |\{g(w) \ : \ l(g) \leq r\}|}{\log r}$$

where $l(g)$ is the length of a group element with respect to some fixed finite generating set of G.

One can show, in the same way as before, that the growth degree γ does not depend on the choice of the generating set of G.

Proposition 5.10. *Suppose that a standard action of a group G on X^* is contracting. Then the growth degree of the action on X^ω is not greater than $\frac{\log |X|}{-\log \rho}$, where ρ is the contraction coefficient of the action on X^*.*

Proof. The statement is more or less classical. See, for instance the similar statements in [Gro81, BG00, Fra70].

Let ρ_1 be such that $\rho < \rho_1 < 1$. Then there exists $C > 0$ and $n \in \mathbb{N}$ such that for all $g \in G$ we have $l(g|_{x_1 x_2 \ldots x_n}) < \rho_1^n \cdot l(g) + C$.

Then cardinality of the set $B(w, r) = \{g(w) \ : \ l(g) \leq r\}$, where $w = x_1 x_2 \ldots \in X^\omega$ is not greater than

$$|X|^n \cdot |\{B\left(x_{n+1} x_{n+2} \ldots, \rho_1^n \cdot r + C\right)|,$$

since the map $\sigma^n : x_1 x_2 \ldots \mapsto x_{n+1} x_{n+2} \ldots$ maps $B(w, r)$ into

$$B\left(x_{n+1} x_{n+2} \ldots, \rho_1^n \cdot r + C\right)$$

and every point of X^ω has exactly $|X|^n$ preimages under σ^n. The map σ^n is the nth iteration of the shift map $\sigma(x_1 x_2 \ldots) = x_2 x_3 \ldots$.

Let $k = \left[\frac{\log r}{-n \log \rho_1}\right] + 1$. Then $\rho_1^{nk} \cdot r < 1$ and the number of the points in the ball $B(w, r)$ is not greater than

$$|X|^{nk} \cdot \left|B\left(\sigma^{nk}(w), R\right)\right|,$$

where

$$R = \rho_1^{nk} \cdot r + \rho_1^{n(k-1)} \cdot C + \rho_1^{n(k-2)} \cdot C + \cdots + \rho_1^n \cdot C + C < 1 + \frac{C}{1 - \rho_1^n}.$$

But $|B(u, R)|$ for all $u \in X^\omega$ is less than $K_1 = |S|^R$, where S is the generating set of G (we assume that $S = S^{-1} \ni 1$). Hence,

$$\begin{aligned} &|B(w, r)| < K_1 \cdot |X|^{n\left(\frac{\log r}{-n \log \rho_1} + 1\right)} = \\ &= K_1 \cdot \exp\left(\frac{\log |X| \log r}{-\log \rho_1} + n \log |X|\right) = K_2 \cdot r^{\frac{\log |X|}{-\log \rho_1}}, \end{aligned}$$

where $K_2 = K_1 \cdot |X|^n$. Thus, the growth degree is not greater than $\frac{\log |X|}{-\log \rho_1}$ for every $\rho_1 \in (\rho, 1)$, so it is not greater than $\frac{\log |X|}{-\log \rho}$. ∎

Lemma 5.11. *Let ϕ be a contracting virtual endomorphism of a ϕ-simple infinite finitely generated group G. Then the contraction coefficient of its standard action is greater or equal to $1/\operatorname{ind}\phi$.*

Proof. Consider the standard action on the set X^* for a standard basis X, containing the element $x_0 = \phi(1)1$. Then the parabolic subgroup $P(\phi) = \cap_{n\geq 0} \operatorname{Dom}\phi^n$ is the stabilizer of the word $w = x_0x_0x_0\ldots \in X^\omega$. The subgroup $P(\phi)$ has infinite index in G, otherwise $\cap_{g\in G} g^{-1}Pg = \mathcal{C}(\phi)$ will have finite index, and G will be not ϕ-simple. Consequently, the G-orbit of w is infinite. Then there exists an infinite sequence of generators $s_1, s_2, \ldots$ of the group G such that the elements of the sequence

$$w, s_1(w), s_2s_1(w), s_3s_2s_1(w), \ldots$$

are pairwise different. This implies that the growth degree of the orbit Gw

$$\gamma = \limsup_{r\to\infty} \frac{|\{g(w) \ : \ l(g) \leq r\}|}{\log r}$$

is greater or equal to 1, thus the growth degree of the action of G on X^ω is not less than 1, and by Proposition 5.10, $1 \leq \frac{\log|X|}{-\log\rho}$. □

Proposition 5.12. *If there exists a faithful contracting action of a finitely-generated group G then for any $\epsilon > 0$ there exists an algorithm of polynomial complexity of degree not greater than $\frac{\log|X|}{-\log\rho} + \epsilon$ solving the word problem in G.*

Proof. We assume that the generating set S is symmetric (i.e., that $S = S^{-1}$) and contains all the restrictions of all its elements, so that always $l(g|_v)$ is not greater than $l(g)$.

We will denote by F the free group generated by S and for every $g \in F$ by $\hat{g}$ we denote the canonical image of g in G.

Let $1 > \rho_1 > \rho$. Then $\rho_1 \cdot |X| > 1$, since by Lemma 5.11, $\rho \cdot |X| \geq 1$. There exist n_0 and l_0 such that for every word $v \in X^*$ of the length n_0 and every $g \in G$ of the length $\geq l_0$ we have

$$l\left(g|_v\right) < \rho_1^n l(g).$$

Assume that we know for every $g \in F$ of the length less than l_0 if $\hat{g}$ is trivial or not. Assume also that we know all the relations $g \cdot v = u \cdot h$ for all $g, l(g) \leq l_0$ and $v \in X^{n_0}$.

Then we can compute in $l(\hat{g})$ steps, for any $g \in F$ and $v \in X^n$, the element $h \in F$ and the word $u \in X^{n_0}$ such that $\hat{g} \cdot v = u \cdot \hat{h}$. If $v \neq u$ then we conclude that $\hat{g}$ is not trivial and stop the algorithm. If for all $v \in X^{n_0}$ we have $v = u$, then $\hat{g}$ is trivial if and only if all the obtained restrictions $\hat{h} = \hat{g}|_v$ are trivial. We know, whether $\hat{h}$ is trivial if $l(h) < l_0$. We proceed further, applying the above computations for those h, which have the length not less than l_0.

But $l(h) < \rho_1^n l(g)$, if $l(g) \geq l_0$. So on each step the length of the elements becomes smaller, and the algorithm stops in not more than $-\log l(g)/\log\rho_1$ steps. On each step the algorithm branches into $|X|$ algorithms. Thus, since $\rho_1 \cdot |X| > 1$, the total time is bounded by

$$
\begin{aligned}
&l(g)\left(1+\rho_1\cdot|X|+(\rho_1\cdot|X|)^2+\cdots+(\rho_1\cdot|X|)^{[-\log l(g)/\log\rho_1]}\right) < \\
&\tfrac{l(g)}{\rho_1\cdot|X|-1}\left((\rho_1\cdot|X|)^{1-\log l(g)/\log\rho_1}-1\right) = \\
&\tfrac{l(g)\rho_1\cdot|X|}{\rho_1\cdot|X|-1}\left((\rho_1\cdot|X|)^{-\log l(g)/\log\rho_1}-(\rho_1\cdot|X|)^{-1}\right) = \\
&C_1 l(g)\left(\exp\left(\log l(g)\left(\tfrac{\log|X|}{-\log\rho_1}-1\right)\right)-C_2\right) = \\
&= C_1 l(g)^{-\log|X|/\log\rho_1} - C_1C_2 l(g),
\end{aligned}
$$

where $C_1 = \frac{\rho_1\cdot|X|}{\rho_1\cdot|X|-1}$ and $C_2 = (\rho_1\cdot|X|)^{-1}$. □

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RESEARCH ARTICLE

Metrizable ball structures

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Abstract. A *ball structure* is a triple (X, P, B), where X, P are nonempty sets and, for any $x \in X$, $\alpha \in P$, $B(x, \alpha)$ is a subset of X, $x \in B(x, \alpha)$, which is called a ball of radius α around x. We characterize up to isomorphism the ball structures related to the metric spaces of different types and groups.

Following [1, 2], by *ball structure* we mean a triple $\mathbf{B} = (X, P, B)$, where X, P are nonempty sets and, for any $x \in X$, $\alpha \in P$, $B(x, \alpha)$ is a subset of X, which is called a ball of radius α around x. It is supposed that $x \in B(x, \alpha)$ for all $x \in X$, $\alpha \in P$.

Let $\mathbf{B}_1 = (X_1, P_1, B_1)$ and $\mathbf{B}_2 = (X_2, P_2, B_2)$ be ball structures, $f : X_1 \to X_2$. We say that f is a $\succ$-*mapping* if, for every $\beta \in P_2$, there exists $\alpha \in P_1$ such that

$$B_2(f(x), \beta) \subseteq f(B_1(x, \alpha))$$

for every $x \in X_1$. If there exists a $\succ$-mapping of X_1 onto X_2, we write $\mathbf{B}_1 \succ \mathbf{B}_2$.

A mapping $f : X_1 \to X_2$ is called a $\prec$-*mapping* if, for every $\alpha \in P_1$, there exists $\beta \in P_2$ such that

$$f(B_1(x, \alpha)) \subseteq B_2(f(x), \beta)$$

for every $x \in X_1$. If there exists an injective $\prec$-mapping of X_1 into X_2, we write $\mathbf{B}_1 \prec \mathbf{B}_2$.

A bijection $f : X_1 \to X_2$ is called an *isomorphism* between $\mathbf{B}_1$ and $\mathbf{B}_2$ if f is a $\succ$-mapping and f is a $\prec$-mapping.

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We say that a property $\mathbf{P}$ of ball structures is a *ball property* if a ball structure $\mathbf{B}$ has a property $\mathbf{P}$ provided that $\mathbf{B}$ is isomorphic to some ball structure with property $\mathbf{P}$.

Example 1. *Let (X, d) be a metric space, $\mathbf{R}^+ = \{x \in \mathbf{R} : x \geq 0\}$. Given any $x \in X$, $r \in \mathbf{R}^+$, put*

$$B_d(x, r) = \{y \in X : d(x, y) \leq r\}.$$

A ball structure $(X, \mathbf{R}^+, B_d)$ is denoted by $\mathbf{B}(X, d)$.

We say that a ball structure $\mathbf{B}$ is *metrizable* if $\mathbf{B}$ is isomorphic to $\mathbf{B}(X, d)$ for some metric space (X, d).

To obtain a characterization (Theorem 1) of metrizable ball structures, we need some definitions and technical results.

A ball structure $\mathbf{B} = (X, P, B)$ is called *connected* if, for any $x, y \in X$, there exists $\alpha \in P$ such that $y \in B(x, \alpha)$, $x \in B(y, \alpha)$.

Lemma 1. *Let $\mathbf{B}_1 = (X_1, P_1, B_1)$ and $\mathbf{B}_2 = (X_2, P_2, B_2)$ be ball structures and let f be a $\prec$-mapping of X_1 onto X_2. If $\mathbf{B}_1$ is connected, then $\mathbf{B}_2$ is connected.*

Proof. Given any $y, z \in X_1$, choose $\alpha \in P_1$ such that $y \in B_1(z, \alpha)$, $z \in B_1(y, \alpha)$. Since f is a $\prec$-mapping, then there exists $\beta \in P_2$ such that $f(B_1(x, \alpha)) \subseteq B_2(f(x), \beta)$ for every $x \in X_1$. Hence, $f(y) \in B_2(f(z), \beta)$ and $f(z) \in B_2(f(y), \beta)$. Since $f(X_1) = X_2$, then $\mathbf{B}_2$ is connected. $\square$

Lemma 2. *Let $\mathbf{B}_1 = (X_1, P_1, B_1)$ and $\mathbf{B}_2 = (X_2, P_2, B_2)$ be ball structures and let f be an injective $\succ$-mapping of X_1 into X_2. If $\mathbf{B}_2$ is connected, then $\mathbf{B}_1$ is connected.*

Proof. Given any $y, z \in X_1$, choose $\beta \in P_2$ such that $f(y) \in B_2(f(z), \beta)$ and $f(z) \in B_2(f(y), \beta)$. Since f is a $\succ$-mapping, then there exists $\alpha \in P_1$ such that $B_2(f(x), \beta) \subseteq f(B_1(x, \alpha))$ for every $x \in X_1$. Since f is injective, then $z \in B_1(y, \alpha)$ and $y \in B_1(z, \alpha)$. Hence, $\mathbf{B}_1$ is connected. $\square$

Let $\mathbf{B} = (X, P, B)$ be a ball structure. For all $x \in X$, $\alpha \in P$, put

$$B^*(x, \alpha) = \{y \in X : x \in B(y, \alpha)\}.$$

A ball structure $\mathbf{B}^* = (X, P, B^*)$ is called *dual* to $\mathbf{B}$. Note that $\mathbf{B}^{**} = \mathbf{B}$.

A ball structure $\mathbf{B}$ is called *symmetric* if the identity mapping $i : X \to X$ is an isomorphism between $\mathbf{B}$ and $\mathbf{B}^*$. In other words, $\mathbf{B}$ is symmetric if, for every $\alpha \in P$, there exists $\beta \in P$ such that $B(x, \alpha) \subseteq B^*(x, \beta)$ for every $x \in X$, and vice versa.

Lemma 3. *Let* $\mathbf{B}_1 = (X_1, P_1, B_1)$ *and* $\mathbf{B}_2 = (X_2, P_2, B_2)$ *be ball structures,* $f : X_1 \to X_2$. *If* f *is a* $\prec$*-mapping of* $\mathbf{B}_1$ *to* $\mathbf{B}_2$*, then* f *is a* $\prec$*-mapping of* $\mathbf{B}_1^*$ *to* $\mathbf{B}_2^*$. *If* f *is an isomorphism between* $\mathbf{B}_1$ *and* $\mathbf{B}_2$*, then* f *is an isomorphism between* $\mathbf{B}_1^*$ *and* $\mathbf{B}_2^*$.

Proof. Let f be a $\prec$-mapping of $\mathbf{B}_1$ to $\mathbf{B}_2$ and let $\alpha \in P_1$. Choose $\beta \in P_2$ such that $f(B_1(x,\alpha)) \subseteq B_2(f(x),\beta)$ for every $x \in X_1$. Take any element $y \in B_1^*(x,\alpha)$. Then $x \in B_1(y,\alpha)$ and $f(x) \in B_2(f(y),\beta)$. Hence, $f(y) \in B_2^*(f(x),\beta)$ and $f(B_1^*(x,\alpha)) \subseteq B_2^*(f(x),\beta)$. It means that f is a $\prec$-mapping of $\mathbf{B}_1^*$ to $\mathbf{B}_2^*$.

Suppose that f is an isomorphism between $\mathbf{B}_1$ and $\mathbf{B}_2$. By the first statement, f is a $\prec$-mapping of $\mathbf{B}_1^*$ to $\mathbf{B}_2^*$ and f^{-1} is a $\prec$-mapping of $\mathbf{B}_2^*$ to $\mathbf{B}_1^*$. It follows that f is an isomorphism between $\mathbf{B}_1^*$ and $\mathbf{B}_2^*$. □

Lemma 4. *Let* $\mathbf{B}_1 = (X_1, P_1, B_1)$ *and* $\mathbf{B}_2 = (X_2, P_2, B_2)$ *be isomorphic ball structures. If* $\mathbf{B}_1$ *is symmetric, then* $\mathbf{B}_2$ *is symmetric.*

Proof. Let $f : X_1 \to X_2$ be an isomorphism between $\mathbf{B}_1$ and $\mathbf{B}_2$. Denote by $i_1 : X_1 \to X_1$ and $i_2 : X_2 \to X_2$ the identity mappings. Clearly, f^{-1} is an isomorphism between $\mathbf{B}_2$ and $\mathbf{B}_1$. By Lemma 3, f is an isomorphism between $\mathbf{B}_1^*$ and $\mathbf{B}_2^*$. By assumption, i_1 is an isomorphism between $\mathbf{B}_1$ and $\mathbf{B}_1^*$. Since $i_2 = f i_1 f^{-1}$, then i_2 is an isomorphism between $\mathbf{B}_2$ and $\mathbf{B}_2^*$. □

A ball structure $\mathbf{B} = (X, P, B)$ is called *multiplicative* if, for any $\alpha, \beta \in P$, there exists $\gamma(\alpha,\beta) \in P$ such that

$$B(B(x,\alpha),\beta) \subseteq B(x,\gamma(\alpha,\beta))$$

for every $x \in X$. Here, $B(A,\alpha) = \bigcup_{a \in A} B(a,\alpha)$ for any $A \subseteq X$, $\alpha \in P$.

Lemma 5. *If a ball structure* $\mathbf{B} = (X, P, B)$ *is multiplicative, then* $\mathbf{B}^*$ *is multiplicative.*

Proof. Given any $\alpha, \beta \in P$, choose $\gamma(\alpha,\beta)$ such that $B(B(x,\alpha),\beta) \subseteq B(x,\gamma(\alpha,\beta))$. Take any element $z \in B^*(B^*(x,\alpha),\beta)$ and pick $y \in B^*(x,\alpha)$ such that $z \in B^*(y,\beta)$. Then $x \in B(y,\alpha)$ and $y \in B(z,\beta)$, so $x \in B(B(z,\beta),\alpha)$. Since $B(B(z,\beta),\alpha) \subseteq B(z,\gamma(\beta,\alpha))$, then $x \in B(z,\gamma(\beta,\alpha))$. Hence, $B^*(B^*(x,\alpha),\beta) \subseteq B^*(x,\gamma(\beta,\alpha))$ and $\mathbf{B}^*$ is multiplicative. □

Lemma 6. *Let* $\mathbf{B}_1 = (X_1, P_1, B_1)$ *and* $\mathbf{B}_2 = (X_2, P_2, B_2)$ *be isomorphic ball structures. If* $\mathbf{B}_1$ *is multiplicative, then* $\mathbf{B}_2$ *is multiplicative.*

Proof. Denote by $f_1 : X_1 \to X_2$ the isomorphism between $\mathbf{B}_1$ and $\mathbf{B}_2$. Fix any $\beta_1, \beta_2 \in P_2$. Since f is a bijection, it suffices to prove that there exists $\beta \in P_2$ such that

$$B_2(B_2(f(x), \beta_1), \beta_2) \subseteq B_2(f(x), \beta)$$

for every $x \in X_1$.

Since f is a $\succ$-mapping, then there exist $\alpha_1, \alpha_2 \in P_1$ such that

$$B_2(f(x), \beta_1) \subseteq f(B_1(x, \alpha_1)), B_2(f(x), \beta_2) \subseteq f(B_1(x, \alpha_2))$$

for every $x \in X_1$.

Since $\mathbf{B}_1$ is multiplicative, then there exists $\alpha \in P_1$ such that

$$B_1(B_1(x, \alpha_1), \alpha_2) \subseteq B_1(x, \alpha)$$

for every $x \in X_1$.

Since f is a $\prec$-mapping, then there exists $\beta \in P_2$ such that

$$f(B_1(x, \alpha)) \subseteq B_2(f(x), \beta)$$

for every $x \in X_1$.

Now fix $x \in X_1$ and take any element $f(z) \in B_2(B_2(f(x), \beta_1), \beta_2)$. Pick $f(y) \in B_2(f(x), \beta_1)$ with $f(z) \in B_2(f(y), \beta_2)$. Then $y \in B_1(x, \alpha_1)$, $z \in B_1(y, \alpha_2)$ and $z \in B_1(B_1(x, \alpha_1), \alpha_2)$. Hence, $z \in B_1(x, \alpha)$ and $f(z) \in B_2(f(x), \beta)$. □

For an arbitrary ball structure $\mathbf{B} = (X, P, B)$, we define a preodering $\leq$ on the set P by the rule

$$\alpha \leq \beta \textit{ if and only if } B(x, \alpha) \subseteq B(x, \beta)$$

for every $x \in X$. A subset P' of P is called *cofinal* if, for every $\alpha \in P$, there exists $\beta \in P'$ such that $\alpha \leq \beta$. A *cofinality* $cf\mathbf{B}$ of $\mathbf{B}$ is a minimum of cardinalities of cofinal subsets of P. Thus, $cf\mathbf{B} \leq \aleph_0$ if and only if there exists a cofinal sequence $< \alpha_n >_{n \in \omega}$ in P such that $\alpha_0 \leq \alpha_1 \leq \ldots \leq \alpha_n \leq \ldots$.

Lemma 7. *If the ball structures* $\mathbf{B}_1 = (X_1, P_1, B_1)$ *and* $\mathbf{B}_2 = (X_2, P_2, B_2)$ *are isomorphic, then* $cf\mathbf{B}_1 = cf\mathbf{B}_2$.

Proof. Let $f : X_1 \to X_2$ be an isomorphism between $\mathbf{B}_1$ and $\mathbf{B}_2$ and let P_1' be a cofinal subset of P_1. Since f is a $\succ$-mapping, then there exists a mapping $h_1 : P_2 \to P_1'$ such that $B_2(f(x), \beta) \subseteq f(B_1(x, h_1(\beta)))$ for any $x \in X_1$, $\beta \in P_2$. Since f is a $\prec$-mapping, then there exists a mapping $h_2 : P_1' \to P_2$ such that $f(B_1(x, \alpha)) \subseteq B_2(f(x), h_2(\alpha))$ for any $x \in X_1$, $\alpha \in P_1'$. From the construction of h_1, h_2 we conclude that $h_2(P_1')$ is a cofinal subset of P_2. Hence, $cf\mathbf{B}_2 \leq cf\mathbf{B}_1$. □

Theorem 1. *A ball structure* $\mathbf{B} = (X, P, B)$ *is metrizable if and only if* $\mathbf{B}$ *is connected symmetric multiplicative and* $cf\mathbf{B} \leq \aleph_0$.

Proof. First suppose that $\mathbf{B}$ is isomorphic to $\mathbf{B}(X, d)$ for an appropriate metric space (X, d). Obviously, $\mathbf{B}(X, d)$ is connected symmetric multiplicative and $cf\mathbf{B} \leq \aleph_0$. By Lemma 1, 4, 6, 7 $\mathbf{B}$ has the same properties.

Now assume that $\mathbf{B}$ is connected symmetric multiplicative and $cf\mathbf{B} \leq \aleph_0$. Let $< \alpha_n >_{n\in\omega}$ be a cofinal sequence in P. Put $\beta_0 = \alpha_0$ and choose $\beta_1 \in P$ such that $\beta_1 \geq \alpha_1$, $\beta_1 \geq \beta_0$, $\beta_1 \geq \gamma(\beta_0, \beta_0)$, where γ is a function from definition of multiplicativity. Suppose that the elements $\beta_0, \beta_1, \dots, \beta_n$ have been chosen. Take $\beta_{n+1} \in P$ such that

$$\beta_{n+1} \geq \alpha_{n+1}, \beta_{n+1} \geq \beta_n, \beta_{n+1} \geq \gamma(\beta_i, \beta_j)$$

for all $i, j \in \{0, 1, \dots, n\}$. Then $< \beta_n >_{n\in\omega}$ is a nondecreasing cofinal sequence in P and $B(B(x, \beta_n), \beta_m) \subseteq B(x, \beta_{n+m})$ for all $x \in X$, $n, m \in \mathbf{N}$.

Define a mapping $d : X \times X \to \omega$ by the rule $d(x, x) = 0$ and

$$d(x, y) = min\{n \in \mathbf{N} : y \in B(x, \beta_n), x \in B(y, \beta_n)\}$$

for all distinct elements $x, y \in X$. Since the sequence $< \beta_n >_{n\in\omega}$ is cofinal in P and $\mathbf{B}$ is connected, then the mapping d is well defined. To show that d is a metric we have only to check a triangle inequality. Let x, y, z be distinct elements of X and let $d(x, y) = n$, $d(y, z) = m$. Since $y \in B(x, \beta_n)$ and $z \in B(y, \beta_m)$, then $z \in B(B(x, \beta_n), \beta_m) \subseteq B(x, \beta_{n+m})$. Since $y \in B(z, \beta_m)$ and $x \in B(y, \beta_n)$, then $x \in B(B(z, \beta_m), \beta_n) \subseteq B(z, \beta_{n+m})$. Hence, $d(x, z) \leq n + m$.

Consider the ball structure $\mathbf{B}(X, d)$ and note that

$$B_d(x, n) = B(x, \beta_n) \bigcap B^*(x, \beta_n).$$

Since $\mathbf{B}$ is symmetric, then the identity mapping of X is an isomorphism between $\mathbf{B}$ and $\mathbf{B}(X, d)$. □

Remark 1. *A metric* d *on a set* X *is called integer if* $d(x, y)$ *is an integer number for all* $x, y \in X$. *It follows from the proof of Theorem 1 that, for every metrizable ball structure* $\mathbf{B} = (X, P, B)$, *there exists an integer metric* d *on* X *such that* $\mathbf{B}$ *and* $\mathbf{B}(X, d)$ *are isomorphic.*

Remark 2. *Let* $\mathbf{B} = (X, P, B)$ *be an arbitrary ball structure. Consider a metric* d *on* X *defined by the rule* $d(x, x) = 0$ *and* $d(x, y) = 1$ *for all distinct elements of* X. *Then the identity mapping* $i : X \to X$ *is a* $\prec$*-mapping of* $\mathbf{B}$ *onto* $\mathbf{B}(X, d)$. *In particular, for every ball structure* $\mathbf{B}$, *there exists a metric space* (X, d) *such that* $\mathbf{B} \prec \mathbf{B}(X, d)$.

Remark 3. *Let $\mathbf{B} = (X, P, B)$ be a connected multiplicative ball structure, $cf\mathbf{B} \leq \aleph_0$. Repeating arguments of Theorem 1, we can prove that there exists a metric d on X such that the identity mapping $i : X \to X$ is a $\prec$-mapping of $\mathbf{B}(X, d)$ onto $\mathbf{B}$.*

Question 1. *Characterize the ball structure $\mathbf{B} = (X, P, B)$, which admit a metric d on X such that the identity mapping $i : X \to X$ is a $\prec$-mapping of $\mathbf{B}(X, d)$ onto $\mathbf{B}$.*

By Remark 2, every ball structure can be strengthened to some mertizable ball structure, so Question 1 asks about ball structure, which can be weekened to metrizable.

Example 2. *Let $Gr = (V, E)$ be a connected graph with a set of vertices V and a set of edges E, $E \subseteq V \times V$. Endow V with a path metric d, where $d(x, y)$, $x, y \in V$ is a length of the shortest path between x and y. Denote by $\mathbf{B}(Gr)$ the ball structure $\mathbf{B}(V, d)$. Obviously, $\mathbf{B}(Gr)$ is metrizable.*

Our next target is a description of the ball structures, isomorphic to $\mathbf{B}(Gr)$ for an appropriate graph Gr.

Let $\mathbf{B} = (X, P, B)$ be an arbitrary ball structure, $\alpha \in P$. We say that a finite sequence $x_0, x_1, \ldots, x_n$ of elements of X is an α-path of length n if $x_{i-1} \in B(x_i, \alpha)$, $x_i \in B(x_{i-1}, \alpha)$ for every $i \in \{1, 2, \ldots, n\}$. A ball structure $\mathbf{B}$ is called an *α-path connected* if, for every $\beta \in P$, there exists $\mu(\beta) \in \omega$ such that $x \in B(y, \beta)$, $y \in B(x, \beta)$ imply that there exists an α-path of length $\leq \mu(\beta)$ between x and y. Note that $\mathbf{B}(Gr)$ is 1-path connected for every connected graph Gr.

A ball structure $\mathbf{B} = (X, P, B)$ is called *path connected* if $\mathbf{B}$ is α-path connected for some $\alpha \in P$.

Lemma 8. *Let $\mathbf{B}_1 = (X_1, P_1, B_1)$ and $\mathbf{B}_2 = (X_2, P_2, B_2)$ be isomorphic ball structures. If $\mathbf{B}_1$ is path connected, then $\mathbf{B}_2$ path connected.*

Proof. Let $f : X_1 \to X_2$ be an isomorphism between $\mathbf{B}_1$ and $\mathbf{B}_2$. Choose $\alpha \in P_1$ such that $\mathbf{B}_1$ is α-path connected and fix a corresponding mapping $\mu : P_1 \to \omega$. Since f is a $\prec$-mapping, then there exists $\beta \in P_2$ such that

$$f(B_1(x, \alpha)) \subseteq B_2(f(x), \beta)$$

for every $x \in X_1$. Since f is a $\succ$-mapping, then there exists a mapping $h : P_2 \to P_1$ such that

$$B_2(f(x), \lambda) \subseteq f(B_1(x, h(\lambda)))$$

for any $x \in X_1, \lambda \in P_2$.

Fix any $\lambda \in P_2$ and suppose that

$$f(x) \in B_2(f(y), \lambda),\ f(y) \in B_2(f(x), \lambda).$$

Since f is a bijection, then $x \in B_1(y, h(\lambda))$, $y \in B_1(x, h(\lambda))$. Since $\mathbf{B}_1$ is α-path connected, then there exists an α-path $x = x_0, x_1, \dots, x_m = y$ of length $\le \mu(h(\lambda))$. Then $f(x) = f(x_0), f(x_1), \dots, f(x_m) = f(y)$ is a β-path of length $\le \mu(h(\lambda))$ between $f(x)$ and $f(y)$. □

Theorem 2. *For every ball structure* $\mathbf{B}$*, the following statements are equivalent*

(i) $\mathbf{B}$ *is metrizable and path connected;*

(ii) $\mathbf{B}$ *is isomorphic to a ball structure* $\mathbf{B}(Gr)$ *for some connected graph* Gr*.*

Proof. (ii)⇒(i). Clearly, $\mathbf{B}(Gr)$ is metrizable and path connected. Hence, $\mathbf{B}$ is metrizable and path connected by Lemma 8.

(i)⇒(ii). Fix a path connected metric space (X, d) such that $\mathbf{B}$ is isomorphic to $\mathbf{B}(X, d)$. Then there exists $m \in \omega$ such that (X, d) is m-path connected. Consider a graph $Gr = (X, E)$ with the set E of edges defined by the rule

$$(x, y) \in E \textit{ if and only if } x \ne y \textit{ and } d(x, y) \le m.$$

Since $\mathbf{B}(X, d)$ is path connected, then the graph Gr is connected.

Let d' be a path metric on the graph Gr. By assumption, for every $n \in \omega$, there exists $\mu(n) \in \omega$ such that $d(x, y) \le n$ implies that there exists a m-path of length $\le \mu(n)$ in (X, d) between x and y. Hence, $d(x, y) \le n$ implies $d'(x, y) \le \mu(n)$. On the other side, $d'(x, y) \le k$ implies that $d(x, y) \le km$. Therefore, the identity mapping of X is an isomorphism between the ball structures $\mathbf{B}(X, d)$ and $\mathbf{B}(Gr)$. □

Example 3. *Let* $X = \{2^n : n \in \omega\}$, $d(x, y) = |x - y|$ *for any* $x, y \in X$*. By Theorem 2, there are no connected graphs* Gr *such that* $\mathbf{B}(X, d)$ *is isomorphic to* $\mathbf{B}(Gr)$*.*

Example 4. *Let* d *be an euclidean metric on* $\mathbf{R}^n$*. By Theorem 2, there exists a connected graph* $Gr_n = (\mathbf{R}^n, E_n)$ *such that* $\mathbf{B}(\mathbf{R}^n, d)$ *is isomorphic to* $\mathbf{B}(Gr_n)$*.*

By Remark 2, for every ball structure $\mathbf{B} = (X, P, B)$, there exists a connected graph $Gr = (X, E)$, $E = \{(x, y) : x, y \in X, x \ne y\}$ such that the identity mapping $i : X \to X$ is a $\succ$-mapping of $\mathbf{B}(Gr)$ onto $\mathbf{B}$.

Question 2. *Characterize the ball structure, which admit a* $\succ$*-bijection to the ball structure* $\mathbf{B}(Gr)$ *for an appropriate graph* Gr*.*

A metric d on a set X is called *non-Archimedian* if

$$d(x,z) \leq max\{d(x,y), d(y,z)\}$$

for all $x, y, z \in X$. The following definitions will be used to describe the ball structures isomorphic to $\mathbf{B}(X,d)$ for an appropriate non-Archimeian metric space (X,d).

Let $\mathbf{B} = (X,P,B)$ be an arbitrary ball structure, $x \in X$, $\alpha \in P$. We say that a ball $B(x,\alpha)$ is a *cell* if $B(y,\alpha) = B(x,\alpha)$ for every $y \in B(x,\alpha)$. If (X,d) is a non-Archimedian metric space, then each ball $B(x,r)$, $x \in X$, $r \in \mathbf{R}^+$ is a cell.

Given any $x \in X$, $\alpha \in P$, denote

$$B^c(x,\alpha) = \{y \in X: \ there \ exists \ an \ \alpha - path \ between \ x \ and \ y\}.$$

A ball structure $\mathbf{B}^c = (X,P,B^c)$ is called a *cellularization* of $\mathbf{B}$. Note that each ball $B^c(x,\alpha)$ is a cell.

We say that a ball structure $\mathbf{B}$ is *cellular* if the identity mapping $i: X \to X$ is an isomorphism between $\mathbf{B}$ and $\mathbf{B}^c$. In other words, $\mathbf{B}$ is cellular if and only if, for every $\alpha \in P$, there exists $\beta \in P$ such that $B(x,\alpha) \subseteq B^c(x,\beta)$ for every $x \in X$ and, for every $\beta \in P$, there exists $\alpha \in P$ such that $B^c(x,\beta) \subseteq B(x,\alpha)$ for every $x \in X$.

A ball structure $\mathbf{B} = (X,P,B)$ is called *directed* if, for any $\alpha, \beta \in P$, there exists $\gamma \in P$ such that $\alpha \leq \gamma$, $\beta \leq \gamma$.

Lemma 9. *If $\mathbf{B} = (X,P,B)$ is a directed symmetric ball structure, then the identity mapping $i: X \to X$ is a $\prec$-mapping of $\mathbf{B}$ onto $\mathbf{B}^c$.*

Proof. Given any $\alpha \in P$, choose $\beta, \gamma \in P$ such that

$$B(x,\alpha) \subseteq B^*(x,\beta) \subseteq B(x,\gamma)$$

for every $x \in X$. Since $\mathbf{B}$ is directed, we may assume that $\beta \leq \gamma$. Take any element $y \in B(x,\alpha)$. Then $x \in B(y,\beta) \subseteq B(y,\gamma)$. Thus, $y \in B(x,\gamma)$, $x \in B(y,\gamma)$. Hence, there exists a β-path of length ≤ 1 between x and y. It means that $y \in B^c(x,\gamma)$, so $B(x,\alpha) \subseteq B^c(x,\gamma)$. $\square$

Lemma 10. *Let $\mathbf{B}_1 = (X_1,P_1,B_1)$ and $\mathbf{B}_2 = (X_2,P_2,B_2)$ be ball structures. If $f: X_1 \to X_2$ is a $\prec$-mapping of $\mathbf{B}_1$ to $\mathbf{B}_2$, then f is a $\prec$-mapping of $\mathbf{B}_1^c$ to $\mathbf{B}_2^c$. If f is an isomorphism between $\mathbf{B}_1$ and $\mathbf{B}_2$, then f is a isomorphism between $\mathbf{B}_1^c$ and $\mathbf{B}_2^c$.*

Proof. Given any $\alpha \in P_1$, choose $\beta \in P_2$ such that $f(B_1(x,\alpha)) \subseteq B_2(f(x),\beta)$ for every $x \in X$. Take any $y \in B_1^c(x,\alpha)$ and choose an α-path $x = x_0, x_1, \ldots, x_n = y$ between x and y. Then

$$f(x) = f(x_0), f(x_1), \ldots, f(x_n) = f(y)$$

is a β-path between $f(x)$ and $f(y)$. Hence, $f(y) \in B_2^c(f(x), \beta)$ and $f(B_1^c(x, \alpha)) \subseteq B_2^c(f(x), \beta)$ for every $x \in X_1$.

Suppose that f is an isomorphism between $\mathbf{B}_1$ and $\mathbf{B}_2$. By the first statement, f is a $\prec$-mapping of $\mathbf{B}_1^c$ to $\mathbf{B}_2^c$ and f^{-1} is a $\prec$-mapping of $\mathbf{B}_2^c$ to $\mathbf{B}_1^c$. Hence, f is an isomorphism between $\mathbf{B}_1^c$ and $\mathbf{B}_2^c$. □

Lemma 11. *Let* $\mathbf{B}_1 = (X_1, P_1, B_1)$ *and* $\mathbf{B}_2 = (X_2, P_2, B_2)$ *be isomorphic ball structures. If* $\mathbf{B}_1$ *is cellular, then* $\mathbf{B}_2$ *is cellular.*

Proof. Let $f : X_1 \to X_2$ be an isomorphism between $\mathbf{B}_1$ and $\mathbf{B}_2$. Denote by $i_1 : X_1 \to X_1$ and $i_2 : X_2 \to X_2$ the identity mappings. Clearly, f^{-1} is an isomorphism between $\mathbf{B}_2$ and $\mathbf{B}_1$. By the Lemma 10, f is an isomorphism between $\mathbf{B}_1^c$ and $\mathbf{B}_2^c$. By assumption, i_1 is an isomorphism between $\mathbf{B}_1$ and $\mathbf{B}_1^c$. Since $i_2 = f i_1 f^{-1}$, then i_2 is an isomorphism between $\mathbf{B}_2$ and $\mathbf{B}_2^c$. □

Theorem 3. *For every ball structure* $\mathbf{B}$*, the following statements are equivalent*

(i) $\mathbf{B}$ *is metrizable and cellular;*

(ii) there exists a non-Archimedian metric space (X,d) such that $\mathbf{B}$ *is isomorphic to* $\mathbf{B}(X, d)$.

Proof. (ii)⇒(i). Clearly, $\mathbf{B}(X, d)$ is metrizable and cellular. Hence, $\mathbf{B}$ is metrizable and cellular by Lemma 11.

(i)⇒(ii). Fix a metric space (X, d') such that $\mathbf{B}(X, d')$ is cellular and isomorphic to $\mathbf{B}$. Define a mapping $d : X \times X \to \omega$ by the rule

$$d(x, y) = min\{m \in \omega : y \in B^c(x, m)\}.$$

Obviously, $d(x, x) = 0$ and $d(x, y) = d(y, x)$ for all $x, y \in X$.

Let $x, y, z \in X$ and let $d(x, y) = m$, $d(y, z) = n$, $m \le n$. Then $y \in B^c(x, m)$, $z \in B^c(y, n)$. It follows that there exists a n-path between x and z. Hence, $z \in B^c(x, n)$ and $d(x, z) \le n$. Thus, we have proved that d is a non-Archimedian metric on X.

Since $d(x, y) \le d'(x, y)$, then the identity mapping $i : X \to X$ is a $\prec$-mapping of $\mathbf{B}(X, d)$ to $\mathbf{B}(X, d')$. Since $\mathbf{B}(X, d')$ is cellular, then there exists a mapping $h : \omega \to \omega$ such that $B^c(x, m) \subseteq B(x, h(m))$ for all $x \in X, m \in \omega$. Hence, i is a $\succ$-mapping of $\mathbf{B}(X, d)$ to $\mathbf{B}(X, d')$. Hence, $\mathbf{B}(X, d)$ and $\mathbf{B}(X, d')$ are isomorphic. □

By Remark 2, for every ball structure $\mathbf{B} = (X, P, B)$, there exists a non-Archimedian metric d on X such that the identity mapping of X is a $\succ$-mapping of $\mathbf{B}(X, d)$ to $\mathbf{B}$.

Lemma 12. *For every metric space (X, d), there exists a family $\{\mathcal{P}_n : n \in \omega\}$ of partitions of X with the following properties*

(i) every partition $\mathcal{P}_{n+1}$ is an enlargement of $\mathcal{P}_n$, i.e. every cell of the partition $\mathcal{P}_{n+1}$ is a union of some cells of the partition $\mathcal{P}_n$;

(ii) there exists a function $f : \omega \to \omega$ such that, for every $C \in \mathcal{P}_n$ and every $x \in C$, $C \subseteq B(x, f(n))$;

(iii) for any $x, y \in X$, there exists $n \in \omega$ such that x, y are in the same cell of the partition $\mathcal{P}_n$.

Proof. Fix any well-ordering $\{x_\alpha : \alpha < \gamma\}$ of X. Choose a subset $Y_0 \subseteq X$, $x_0 \in Y_0$ such that the family $\{B(y, 1) : y \in Y_0\}$ is disjoint and maximal. For every $x \in X$, pick a minimal element $f_0(x) \in Y_0$ such that $B(x, 1) \bigcap B(f_0(x), 1) \neq \emptyset$. Put $H(x, 1) = \{z \in X : f_0(z) = f_0(x)\}$ and note that the family $\{H(y, 1) : y \in Y_0\}$ is a partition of X. If $x, z \in H(y, 1)$, then $d(x, y) \leq 2$, $d(x, z) \leq 2$. Therefore, $H(y, 1) \subseteq B(x, 4)$ for every $x \in H(y, 1)$. Put $\mathcal{P}_0 = \{H(y, 1) : y \in Y_0\}$, $f(0) = 4$.

Assume that the partitions $\mathcal{P}_0, \mathcal{P}_1, \dots, \mathcal{P}_{n-1}$ have been constructed and the values $f(0)$, $f(1)$, …, $f(n-1)$ have been determined. Choose a subset $Y_n \subseteq X$, $x_0 \in Y_n$ such that the family $\{B(y, n+1) : y \in Y_n\}$ is disjoint and maximal. Define a mapping $f_n : X \to Y_n$ inductively such that f_n is constant on each cell of the partition $\mathcal{P}_{n-1}$. Put $f_n(x) = x_0$ for every $x \in X$ such that $H(x, n) \bigcap B(x_0, n+1) \neq \emptyset$. Then take the minimal element $x \in X$ such that $f_n(x)$ is not determined. Choose the minimal element $y \in Y_n$ such that $B(x, n+1) \bigcap B(y, n+1) \neq \emptyset$. Put $f_n(x) = y$ and $f_W(z) = y$ for every $z \in H(x, n)$. After this transfinite procedure, we denote $H(x, n+1) = \{z \in X : f_n(z) = f_n(x)\}$. Put $\mathcal{P}_n = \{H(y, n+1) : y \in Y_n\}$. Then $\mathcal{P}_n$ is a partition of X and each cell of $\mathcal{P}_n$ is a union of some cells of $\mathcal{P}_{n-1}$. Thus, (i) is satisfied.

If $z \in H(y, n+1)$, then $d(z, y) \leq f(n-1) + 2(n+1)$. Hence, to satisfy (ii), put $f(n) = 2(f(n-1) + 2(n+1))$.

At last, given any $x, y \in X$, choose $m \in \omega$ such that $d(x_0, x) \leq m+1$, $d(x_0, y) \leq m+1$. Thus x, y are in the same cell of the partition $\mathcal{P}_m$ and we have verified (iii). □

Theorem 4. *For every metric space (X,d), there exists a non-Archimedian metric $d^{'}$ on X such that the identity mapping $i : X \to X$ is a $\prec$-mapping of $\mathbf{B}(X, d^{'})$ to $\mathbf{B}(X, d)$.*

Proof. Fix a family $\{\mathcal{P}_n : n \in \omega\}$ of partitions of X, satisfying (i), (ii), (iii) from Lemma 12. Define a mapping $d^{'} : X \times X \to \omega$ by the rule

$$d^{'}(x, y) = min\{n : x \text{ and } y \text{ are in the same cell of } \mathcal{P}_n\}.$$

By (iii), $d^{'}$ is well defined. By (i), $d^{'}$ is a non-Archimedian metric. By (ii), the identity mapping of X is a $\prec$-mapping of $\mathbf{B}(X, d^{'})$ onto $\mathbf{B}(X, d)$. $\square$

Now we consider non-metrizable versions of Lemma 12 and Theorem 4.

Lemma 13. *Let $\mathbf{B} = (X, P, B)$ be a directed symmetric multiplicative ball structure. Then there exists a family $\{\mathcal{P}_\alpha : \alpha \in P\}$ of partitions of X such that*

(i) for every $\alpha \in P$, there exists $\beta \in P$ such that $C \subseteq B(x, \beta)$ for every $C \in \mathcal{P}_\alpha$ and every $x \in C$.

Moreover, if $\mathbf{B}$ is connected then

(ii) for any $x, y \in X$, there exists $\alpha \in P$ such that x, y are in the same cell of the partition $\mathcal{P}_\alpha$.

Proof. Fix any well-ordering of X and denote by x_0 its minimal element. Fix $\alpha \in P$ and choose a subset $Y \subseteq X$, $x_0 \in Y$ such that the family $\{B(y, \alpha) : y \in Y\}$ is disjoint and maximal. For every $x \in X$, pick a minimal element $f(x) \in Y$ such that $B(x, \alpha) \bigcap B(f(x), \alpha) \neq \emptyset$. Put $H(x, \alpha) = \{z \in X : f(z) = f(x)\}$. Then the family $\mathcal{P}_\alpha = \{H(y, \alpha) : y \in Y\}$ is a partition of X.

Since $\mathbf{B}$ is directed and symmetric, then there exists $\alpha^{'} > \alpha$ such that $y \in B(x, \alpha)$ implies $x \in B(y, \alpha^{'})$.

Fix $x \in X$ and take $x^{'} \in B(x, \alpha) \bigcap B(f(x), \alpha)$. Then x, $x^{'}$, $f(x)$ is an $\alpha^{'}$-path. Hence, for every $z \in H(x, \alpha)$, we can find an $\alpha^{'}$-path of length 4 between x and z. Using multiplicativity of $\mathbf{B}$, choose $\beta \in P$ such that $y_4 \in B(y_0, \beta)$ for every $\alpha^{'}$-path y_0, y_1, y_2, y_3, y_4 in X. Then $H(x, \alpha) \subseteq B(x, \beta)$.

Suppose that $\mathbf{B}$ is connected and $x, y \in X$. Since $\mathbf{B}$ is directed, then there exists $\alpha \in P$ such that $x_0 \in B(x, \alpha)$, $x_0 \in B(y, \alpha)$. Hence, x, y belong to the cell $H(x_0, \alpha)$ of the partition $\mathcal{P}_\alpha$. $\square$

Theorem 5. *If a ball structure $\mathbf{B} = (X, P, B)$ is directed symmetric and multiplicative, then there exists a cellular ball structure $\mathbf{B}^{'} = (X, P, B^{'})$ such that the identity mapping of X is a $\prec$-mapping of $\mathbf{B}^{'}$ onto $\mathbf{B}$. Moreover, if $\mathbf{B}$ is connected, then $\mathbf{B}^{'}$ is connected.*

Proof. Use the family of the partitions $\{\mathcal{P}_\alpha : \alpha \in P\}$ from Lemma 13 and put $B^{'}(x, \alpha) = H(x, \alpha)$. Clearly, each ball $B^{'}(x, \alpha)$ is a cell. By (i), the identity mapping of X is a $\prec$-mapping of $\mathbf{B}^{'}$ onto $\mathbf{B}$. If $\mathbf{B}$ is connected, then $\mathbf{B}^{'}$ is connected by (ii). $\square$

Example 5. *Let G be a group and let $Fin_e(G)$ be a family of all finite subsets of G containing the identity e. Given any $g \in G$, $F \in Fin_e(G)$, put $B(g, F) = Fg$. A ball structure $\mathbf{B}(G) = (G, Fin_e(G), B)$ is denoted by $\mathbf{B}(G)$. It is easy to show, that $\mathbf{B}(G)$ is directed connected symmetric and multiplicative.*

Now we apply the above results to the ball structures of groups.

Theorem 6. *Let G be a group. Then a ball structure $\mathbf{B}(G)$ is metrizable if and only if $|G| \leq \aleph_0$.*

Proof. Apply Theorem 1. □

Theorem 7. *For every group G, the following statements are equivalent*
(i) G is finitely generated;
(ii) $\mathbf{B}(G)$ is isomorphic to $\mathbf{B}(Gr)$ for some connected graph Gr

Proof. (i)⇒(ii). Let S be a finite set of generators of G. Consider a Cayley graph $Gr = (G, E)$ of G determined by S. By definition, $(x, y) \in E$ if and only if $x \neq y$ and $x = ty$ for some $t \in S \bigcup S^{-1}$. Clearly, the identity mapping of G is an isomorphism between $\mathbf{B}(G)$ and $\mathbf{B}(Gr)$.

(ii)⇒(i). By Theorem 2, there exists $F \in Fin$ such that $\mathbf{B}(G)$ is F-path connected. In particular, for every $g \in G$, there exists a F-path between e and g. Hence, F generates G. □

A group G is called *locally finite* if every finite subset of G generates a finite subgroup.

Theorem 8. *Let G be a group. Then a ball structure $\mathbf{B}(G)$ is cellular if and only if G is locally finite.*

Proof. Let G be locally finite. Denote by Fin_s the family of all finite subgroups of G. Then Fin_s is cofinal in Fin and each ball $B(g, F)$, $F \in Fin_s$ is a cell. Hence, $\mathbf{B}(G)$ is cellular.

Assume that $\mathbf{B}(G)$ is cellular. Note that $B^c(e, F) = gpF$ for every $F \in Fin$, where gpF is a subgroup of G generated by F. Since $\mathbf{B}$ is isomorphic to $\mathbf{B}^c$, then each ball $B^c(g, F)$ is finite. In particular, gpF is finite for every $F \in Fin$. □

Remark 4. *Let G_1, G_2 be countable locally finite group. By [2, Theorem 4], $\mathbf{B}(G_1) \succ \mathbf{B}(G_2)$ and $\mathbf{B}(G_1) \prec \mathbf{B}(G_2)$. By [2, Theorem 5], $\mathbf{B}(G_1)$ and $\mathbf{B}(G_2)$ are isomorphic if and only if, for every finite subgroup F of G_1, there exists a finite subgroup H of G_2 such that $|F|$ is a divisor of $|H|$, and vice versa. A problem of classification up to an isomorphism of ball structures of uncountable locally finite groups is open.*

Theorem 9. *For every countable group G, there exists a non-Archimedian metric d on G with the following property*

(i) for each $n \in \omega$, there exists $F \in Fin$ such that $d(x, y) \leq n$ implies $x \in Fy$.

Proof. Apply Theorem 6 and Theorem 4. □

Theorem 10. *For every group G, there exists a cellular ball structure $\mathbf{B}' = (G, Fin, B')$ such that the identity mapping of G is a $\prec$-mapping of $\mathbf{B}'$ onto $\mathbf{B}(G)$.*

Proof. Apply Theorem 5. □

Question 3. *Characterize the ball structures isomorphic to the ball structures of groups.*

M.Zarichnyi has pointed out that Theorem 1 has a counterpart in the asymptotic topology [3]. This theorem answers the Open Question 1 from [4]. The results of this paper was announced in [5].

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