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$\mathcal{H}-$,$\mathcal{R}-$ and $\mathcal{L}-$cross-sections of the infinite symmetric inverse semigroup IS_X

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Abstract. All $\mathcal{H}-$,$\mathcal{R}-$ and $\mathcal{L}-$cross-sections of the infinite symmetric inverse semigroup IS_X are described.

Introduction

Let ρ be an equivalence relation on a semigroup S. The subsemigroup $T \subset S$ is called a *cross-section* with respect to ρ if T contains exactly 1 element from every equivalence class. Clearly, the most interesting are the cross-sections with respect to the equivalence relations connected with the semigroup structure on S. The first candidates for such relations are congruences and the Green relations.

The *Green relations* $\mathcal{L}, \mathcal{R}, \mathcal{H}, \mathcal{D}$ and $\mathcal{J}$ on semigroup S are defined as binary relations in the following way: $a\mathcal{L}b$ if and only if $S^1a = S^1b$; $a\mathcal{R}b$ if and only if $aS^1 = bS^1$; $a\mathcal{J}b$ if and only if $S^1aS^1 = S^1bS^1$ for any $a, b \in S$ and $\mathcal{H} = \mathcal{L} \wedge \mathcal{R}$, $\mathcal{D} = \mathcal{L} \vee \mathcal{R}$.

Cross-sections with respect to the $\mathcal{H} - (\mathcal{L}-, \mathcal{R}-, \mathcal{D}-, \mathcal{J}-)$ Green relations are called $\mathcal{H} - (\mathcal{L}-, \mathcal{R}-, \mathcal{D}-, \mathcal{J}-)$ *cross-sections* in the sequel.

The study of cross-sections with respect to Green relations for some classical semigroups was initiated a few years ago. For the semigroup IS_n all $\mathcal{H}-$cross-sections were classified in [CR] and all $\mathcal{L}-$ and $\mathcal{R}-$cross-sections were classified in [GM1]. For the full transformation semigroup

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$\mathcal{T}_X$ all $\mathcal{H}-$ and $\mathcal{R}-$cross-sections were described in [P1] and [P2], and for the Brauer semigroup all $\mathcal{H}-$,$\mathcal{L}-$ and $\mathcal{R}-$cross-sections were classified in [KMM].

In the present paper all $\mathcal{H}-$,$\mathcal{R}-$ and $\mathcal{L}-$ cross-sections of the infinite symmetric inverse semigroup IS_X are described. The paper is organized as follows. We collect all necessary preliminaries in Section 1. Section 2 is dedicated to the construction and classification of all $\mathcal{H}-$cross-sections of IS_X. Also we prove that every two $\mathcal{H}-$cross-sections are isomorphic. In Section 3 we describe all $\mathcal{R}-$ and $\mathcal{L}-$cross-sections in IS_X. Since infinity of the set X is not used in the proof, we see that from this description one immediately gets the well-known(see [GM1]) description of the $\mathcal{R}-(\mathcal{L}-)$ cross-sections for the finite symmetric inverse semigroup IS_n. Finally, in Section 4 we determine, which $\mathcal{R}-(\mathcal{L}-)$ cross-sections are isomorphic.

1. Preliminaries

Let X be an arbitrary infinite set.

The *symmetric inverse semigroup* on X is the semigroup of all one-to-one partial transformations on X under composition. It is denoted by IS_X. For $a \in IS_X$ by $dom(a)$ and $im(a)$ we denote the domain and the image of the element a respectively. The cardinal number $\mathrm{rk}(a) = |dom(a)| = |im(a)|$ is called the *rank* of a.

It is well-known (see for example [GM2]) that the Green relations on IS_X can be described as follows:

$a\mathcal{R}b$ if and only if $dom(a) = dom(b)$;

$a\mathcal{L}b$ if and only if $im(a) = im(b)$;

$a\mathcal{H}b$ if and only if $dom(a) = dom(b)$ and $im(a) = im(b)$;

$a\mathcal{D}b$ if and only if $\mathrm{rk}(a) = \mathrm{rk}(b)$.

In particular, Green's $\mathcal{D}-$classes are $D_k = \{ a \in IS_X \mid \mathrm{rk}(a) = k\},\ 1 \leq k \leq |X|$.

Recall that a binary relation $<$ on X is a *well order* if it is reflexive, antisymmetric, transitive and satisfies the following properties: **(i)** for all $x, y \in X$, either $x < y$ or $y < x$; **(ii)** every non-empty subset $Y \subseteq X$ has the smallest element.

If the set X is equipped with a well order, then denote by $\xi(X)$ the order-type of this ordered set. Denote by $W(\alpha)$ the set of all ordinal numbers less than α. If $\xi(X) = \alpha$, then there exists a unique isomorphism $f : X \to W(\alpha)$. Denote by $x_\beta := f^{-1}(\beta)$ for every $\beta \in W(\alpha)$. Then $X = \bigcup_{\beta<\alpha}\{x_\beta\}$, moreover, $x_\beta < x_\gamma$ iff $\beta < \gamma$. For every $\eta \leq \alpha$ denote by $X(\eta)$ the set $\{x_\beta \in X | \beta < \eta\}$.

Let ω be the order-type of the natural numbers in their usual order.

2. Description of $\mathcal{H}-$ cross-sections

From the structure of Green relation $\mathcal{H}$ on the semigroup IS_X it follows that each $\mathcal{H}-$class of this semigroup is uniquely determined by two sets $A, B \subseteq X$ with $|A| = |B|$. Denote by $H(A, B)$ the $\mathcal{H}-$class determined by these sets.

Theorem 1. *Let X be an countable set and $<$ be an arbitrary well order of type ω on the set X. Then*

$$I(X,<) = \{a \in IS_X|\ x < y \text{ implies } a(x) < a(y) \text{ for all } x, y \in dom(a)\}$$

is an $\mathcal{H}-$cross-section of IS_X.

Moreover, if $<_1 \neq <_2$, then $I(X,<_1) \neq I(X,<_2)$.

Proof. It is obvious, that $I(X,<)$ is closed under multiplication. Also, since ω is the smallest transfinite number, we see that for every $\mathcal{H}-$class H the intersection $H \cap I(X,<)$ contains exactly one element. This completes the proof of the first part of our theorem.

Let $x'_1 <_1 x'_2 <_1 x'_3 <_1 \ldots$ and $x''_1 <_2 x''_2 <_2 x''_3 <_2 \ldots$ be two different well orders of the type ω on the set X. By k denote the smallest number such that $x'_k \neq x''_k$. We consider the following two cases:

1) $k = 1$. Let $x''_m = x'_1 = x, x'_n = x''_1 = y$. Then the set $Y := \{x \in X | x >_1 x'_n, x >_2 x''_m\}$ is not empty. By z denote an arbitrary element of Y. Then $x <_1 y <_1 z$ and $y <_2 x <_2 z$. Therefore in this case, we have $\begin{pmatrix} x & y \\ y & z \end{pmatrix} \in I(X,<_1)$ and also $\begin{pmatrix} x & y \\ y & z \end{pmatrix} \notin I(X,<_2)$. Hence, $I(X,<_1) \neq I(X,<_2)$.

2) $k > 1$. Let $x'_1 = x, x'_k = y, x''_k = z$. Then $x <_1 y <_1 z$ and $x <_2 z <_2 y$. Arguing as above, we see that $I(X,<_1) \neq I(X,<_2)$. □

Theorem 2. *Suppose X is an arbitrary infinite set.*

a) The semigroup IS_X contains $\mathcal{H}-$cross-sections if and only if the set X is countable.

b) If X is countable, then every $\mathcal{H}-$cross-section of IS_X has the form $I(X,<)$ for some well order $<$ of the type ω on the set X. Moreover, every two $\mathcal{H}-$cross-sections are isomorphic.

Proof. a) *Sufficiency* follows from Theorem 1.

Necessity. Let T be an $\mathcal{H}-$cross-section of IS_X. Let K denote the complete graph on X. We orient the edges E of K as follows:

For any $x, y \in X$, let $a_{x,y}$ be a unique element of T such that $a_{x,y}(\{1,2\}) = \{x, y\}$. Note that $a_{x,y} = a_{y,x}$. Define

$$(x,y) \in E \quad \text{if } a_{x,y}(1) = x, a_{x,y}(2) = y$$
$$(y,x) \in E \quad \text{if } a_{x,y}(1) = y, a_{x,y}(2) = x.$$

Clearly this provides an orientation of the edges.

We proceed by a sequence of lemmas.

Lemma 1. *Let a be an arbitrary element of T and $x, y \in dom(a)$. If $(x, y) \in E$, then $(a(x), a(y)) \in E$.*

Proof. The proof is analogous to one of Lemma 3.3 in [CR]. □

Lemma 2. *K has no cycles.*

Proof. The proof is analogous to one of Lemma 3.4 in [CR]. □

Lemma 3. *K does not contain two infinite paths such that one of them possesses an initial vertex and the other possesses a terminal vertex.*

Proof. Assume the converse. Let $(x_1, x_2, \cdots)$ and $(\cdots, y_{-1}, y_0)$ be two such paths. Suppose a is a unique element of the set $T \cap H(\{x_i | i \in \mathbb{N}\}, \{y_i | i \in \mathbb{Z} \setminus \mathbb{N}\})$. Let $y_k = a(x_1)$ and $x_l = a^{-1}(y_{k-1})$. Then by Lemma 1, we obtain $(x_l, x_0) \in E$. This contradicts Lemma 2 and the lemma is proved. □

From the previous Lemma it follows that K does not contain two-sided infinite paths.

We define the graph $K' = (X, E')$ as follows:

$$\begin{array}{ll} K' = K, & \text{if every infinite path of } K \text{ possesses an initial vertex,} \\ K' = K^c, & \text{if every infinite path of } K \text{ possesses a terminal vertex,} \end{array}$$

where K^c has the same vertex set as K and an arrow is in K^c if and only if its converse is in K. Then every infinite path of the graph K' possesses an initial vertex and also Lemmas 1-3 hold true for this graph.

For arbitrary $x \in X$ denote by P_x the set $\{y \in X | (y, x) \in E'\}$.

Lemma 4. *If $|P_x| > 0$, then there exists a unique element $x_p \in P_x$ such that $(y, x_p) \in E'$ for all $y \in P_x \setminus \{x_p\}$.*

Proof. Consider an element x_1 of P_x. Let us move along the arrows of K' the end of which also belongs to P_x. Assume this process is infinite. Then since K' has no cycles, we obtain an infinite path $(x_1, x_2, x_3, \cdots)$. Moreover, $x_i \in P_x$ for all $i \in \mathbb{N}$. Suppose a is a unique element of the set $T \cap H(\{x_i | i \in \mathbb{N}\} \cup \{x\}, \{x_i | i \in \mathbb{N}\})$. Let $x_k = a(x)$ and $x_l = a^{-1}(x_{k+1})$. Then by Lemma 1, we have $(x, x_l) \in E'$. This contradicts $x_l \in P_x$. This implies that there exists a finite path $(x_1, x_2, \ldots, x_n)$ which it is impossible to prolong. Thus there is no arrow with the beginning x_n and the end in P_x. Hence x_n satisfies lemma's conditions. Now assume there exists an element y with the above property. Then $(x_n, y) \in E'$,$(y, x_n) \in E'$ and by Lemma 2 $y = x_n$. This completes the proof of the lemma. □

Lemma 5. *For any non-empty subset $Y \subseteq X$ there exists a unique element $z \in Y$ such that $(z, y) \in E'$ for all $y \in Y \backslash \{z\}$.*

Proof. Assume there is no such an element. Since K' has no cycles, we see that starting at any element of the set Y we can move opposite the direction of the arrows infinitely long and thus construct an infinite path without an initial vertex. This contradicts the definition of the graph K'. Now assume there exist two different elements x and z with the above property. Then $(x, z) \in E'$ and $(z, x) \in E'$. This contradicts Lemma 2 and the statement is proved. □

Define $x < y \iff$ (either $x = y$ or $(x, y) \in E'$).

Lemma 6. *The relation $<$ is a linear order.*

Proof. From the definition of the graph K' it follows that either $x < y$ or $y < x$ for all $x, y \in X$. Now let $x < y$ and $y < z$. Assume that $z < x$; then the graph K' contains the cycle $x - y - z - x$. This contradicts Lemma 2 and so $x < z$. Thus $<$ is transitive. Since the proof of reflexivity and anti-symmetry of the relation are trivial, the lemma is proved. □

Lemma 7. *The relation $<$ is a well order of type ω.*

Proof. From Lemma 5 and Lemma 6 it follows that the order $<$ is a well order. Also we have that for any $x \in X$ such that $|P_x| > 0$ there exists a predecessor by Lemma 4. This completes the proof of the lemma. □

By Lemma 7 the set X is countable.

b) Suppose X is an arbitrary countable set, T is an $\mathcal{H}-$cross-section of IS_X, and $<$ is the well order of the type ω on X defined in the proof of item a). Let $S = I(X, <)$. From Lemma 1, we see that $T \subseteq S$. Since T and S contain exactly one element from every $\mathcal{H}-$class of IS_X, we obtain $T = S$.

Now let $S_1 = I(X, <_1)$ and $S_2 = I(X, <_2)$ be two $\mathcal{H}-$cross-sections of IS_X determined by the orders

$$x'_1 <_1 x'_2 <_1 x'_3 <_1 \ldots$$

$$x''_1 <_2 x''_2 <_2 x''_3 <_2 \ldots$$

Let θ denote the permutation of X such that $x'_i \mapsto x''_i$ $(i \in \mathbb{N})$. Then the mapping

$$\Theta : \alpha \mapsto \theta^{-1}\alpha\theta \ (\alpha \in S_1)$$

is an isomorphism of S_1 onto S_2. □

3. Description of $\mathcal{R}-$ and $\mathcal{L}-$ cross-sections

Since for $a, b \in IS_X$ the condition $a\mathcal{R}b$ is equivalent to the condition $dom(a) = dom(b)$, the equalities $a = b$ and $dom(a) = dom(b)$ are equivalent for elements a, b from arbitrary $\mathcal{R}-$cross-section T of IS_X. We will frequently use this fact in the paper.

>From the structure of Green relation $\mathcal{R}$ on the semigroup IS_X it follows that each $\mathcal{R}-$class of this semigroup is uniquely determined by a set $A \subseteq X$. Denote by $R(A)$ the $\mathcal{R}-$class determined by this set.

Let a well order $<$ on the set X be fixed and $\xi(X) = \alpha$.

Now construct the set $R(X, <)$ in the following way: an element $a \in R(A)$ with $\xi(A) = \eta \leq \alpha$ belongs to $R(X, <)$ if and only if the map a is an isomorphism of the well-ordered sets A and $X(\eta)$. Then it is obvious, that $R(X, <)$ contains exactly one element from every $\mathcal{R}-$class.

Lemma 8. *For every well order $<$ on the set X the set $R(X, <)$ is closed under multiplication.*

Proof. Let $a, b \in R(X, <)$ be arbitrary elements. Then there exist two $\mathcal{R}-$classes $R(A)$, $R(B)$ such that $a \in R(A), b \in R(B)$. Let us give some notation.

$$\eta_A := \xi(A), \ \eta_B := \xi(B), \ C := X(\eta_A) \cap B, \ \eta_C := \xi(C) = \xi(b|_C).$$

Then $dom(ab) = a^{-1}(C)$ and $\xi(dom(ab)) = \eta_C$. To complete the proof it is now enough to show that $b|_C = X(\eta_C)$. First suppose $X(\eta_C) \nsubseteq b|_C$. Let x_γ be the smallest element of the set $X(\eta_C)\backslash b|_C$. Since $\gamma < \eta_C$, there exists $\delta > \gamma$ such that $x_\delta \in b|_C$, because otherwise $b|_C \subset X(\gamma)$ and this implies $\eta_C = \xi(b|_C) \leq \gamma$. Moreover, from $\gamma < \eta_C$ it follows that $\gamma < \eta_B$ and $x_\gamma \in im(b)$. Let $x_\epsilon := b^{-1}(x_\gamma)$. Since $x_\epsilon \notin C$, we have $x_\epsilon \geq x_\alpha > b^{-1}(x_\delta)$. This contradicts to the fact that b is an isomorphism of well-ordered sets. Thus our assumption is wrong. Therefore $X(\eta_C) \subseteq b|_C$. Now the equality $b|_C = X(\eta_C)$ immediately follows from $\xi(b|_C) = \eta_C$. □

Lemma 9. *For every well order $<$ on the set X the set $R(X, <)$ is an $\mathcal{R}-$cross-section in IS_X.*

Proof. By Lemma 8 this set is closed under multiplication. Hence $R(X, <)$ is a subsemigroup of IS_X. But from the construction of this set it also follows that $R(X, <)$ contains exactly one element from every $\mathcal{R}-$class and the statement is proved. □

Let now $X = \bigcup_{i\in I} X_i$ be an arbitrary decomposition of X into a disjoint union of non-empty blocks, where the order of blocks is not important. Assume that a well order $<_i$ is fixed on the elements of the block X_i for all $i \in I$. The decomposition $X = \bigcup_{i\in I} X_i$ together with a fixed well order on every block will be denoted by $\{\bigcup_{i\in I}(X_i, <_i)\}$. The notation $\{\bigcup_{i\in I}(X_i, <_i)\} \neq \{\bigcup_{j\in J}(X_j, <_j)\}$ then means that either the decompositions $X = \bigcup_{i\in I} X_i$ and $X = \bigcup_{j\in J} X_j$ are different or there exists a block on which the fixed well orders are different.

Let α_i be the order-type of the set X_i. Now construct the set $R(\{\bigcup_{i\in I}(X_i, <_i)\})$ in the following way: an element $a \in R(A)$ belongs to $R(\{\bigcup_{i\in I}(X_i, <_i)\})$ if and only if the map $a|_{A\cap X_i}$ is an isomorphism of $A \cap X_i$ and $X_i(\eta_i)$, where $\eta_i = \xi(A \cap X_i) \leq \alpha_i$ for all $i \in I$.

Theorem 3. *a) For an arbitrary decomposition $X = \bigcup_{i\in I} X_i$ and arbitrary well orders on the elements of every block of this decomposition the set $R(\{\bigcup_{i\in I}(X_i, <_i)\})$ is an $\mathcal{R}-$cross-section of IS_X.*

b) If $\{\bigcup_{i\in I}(X_i, <_i)\} \neq \{\bigcup_{j\in J}(X_j, <_j)\}$ then one has that $R(\{\bigcup_{i\in I}(X_i, <_i)\}) \neq R(\{\bigcup_{j\in J}(X_j, <_j)\})$.

c) Moreover, every $\mathcal{R}-$cross-section of IS_X has the form $R(\{\bigcup_{i\in I}(X_i, <_i)\})$ for some decomposition $X = \bigcup_{i\in I} X_i$ and some well orders $<_i$ on the elements of every block.

Proof. a) We can regard elements of $R(\{\bigcup_{i\in I}(X_i, <_i)\})$ as all possible collections $(a_i \in R(X_i, <_i))_{i\in I}$ with component-wise multiplication. Therefore, the item a) follows from Lemma 9.

b) Obvious.

c) Now let T be an $\mathcal{R}-$cross-section of IS_X. By I denote the set $\{x \in X|\ id_{\{x\}} \in T\}$. By definition, put $X_i = \{a^{-1}(i)|\ a \in T$ and $im(a) = \{i\}\}$. We consider the following two cases:

Case 1. $|I| = 1$. Let $I = \{x_0\}$. Denote by P the set $\{im(a)|a \in T\}$. To prove the theorem, we need several lemmas.

Lemma 10. *For all $A, B \in P$ we have either $A \subseteq B$ or $B \subseteq A$.*

Proof. Assume the converse. Then there exist $A, B \in P$ such that $B\backslash A \neq \emptyset$ and $A\backslash B \neq \emptyset$. Let $y \in B\backslash A$, $z \in A\backslash B$. Choose an element $a \in T$ such that $im(a) = A$ and an element $b \in T$ such that $im(b) = B$. Denote by c a unique element of the set $T \cap R(\{y, z\})$. Then $im(ac) = \{c(z)\}$. Since $ac \in D_1$, we have $im(ac) = \{x_0\}$. Thus $c(z) = x_0$. One can similarly prove that $c(y) = x_0$. This contradicts the injectivity of c and completes the proof. $\square$

Lemma 11. *Let $k \in \mathbb{N}$ and $a, b \in D_k \cap T$. Then $im(a) = im(b)$.*

Proof. Follows from the previous lemma. □

For any natural number k by M_k denote the set $im(D_k \cap T)$. It follows from Lemma 10 that $M_k \subset M_{k+1}$ for all $k \in \mathbb{N}$. Therefore, $|M_{k+1} \backslash M_k| = 1$. Denote by x_k a unique element of the set $M_{k+1} \backslash M_k$. Then $M_k = \{x_0, x_1, \ldots, x_{k-1}\}$.

We construct the relation $<$ as follows:

Define $x < x$ for all $x \in X$. For any $x, y \in X$ such that $x \neq y$, let $a_{x,y}$ be a unique element of the set $T \cap R(\{x, y\})$. Note that $a_{x,y} = a_{y,x}$ and $im(a_{x,y}) = \{x_0, x_1\}$. Define

$$\begin{array}{ll} x < y & \text{if } a_{x,y}(x) = x_0, a_{x,y}(y) = x_1 \\ y < x & \text{if } a_{x,y}(y) = x_0, a_{x,y}(x) = x_1. \end{array}$$

Lemma 12. *Let a be an arbitrary element of T and $x, y \in dom(a)$. If $x < y$, then $a(x) < a(y)$.*

Proof. Let $x' = a(x), y' = a(y)$ and $b = a_{x',y'}$. Since $dom(ab) = \{x, y\} = dom(a_{x,y})$, we have $ab = a_{x,y}$. This implies $(ab)(x) = a_{x,y}(x)$. Also, since $x < y$, we obtain $b(x') = b(a(x)) = (ab)(x) = a_{x,y}(x) = x_0$. Finally, from the definition of $<$ it follows that $x' < y'$, that is, $a(x) < a(y)$. □

Lemma 13. *The relation $<$ is a linear order.*

Proof. From the definition of $<$ it follows that for all different $x, y \in X$ we have either $x < y$ or $y < x$. Reflexivity and anti-symmetry of the relation are obvious. Therefore, to complete the proof it is now enough to prove the transitivity. Considering the product ba_{x_k,x_l}, where $b \in T \cap D_{k+1}$, we obtain $x_k < x_l$ for all natural numbers $k < l$. Suppose x, y, z are three different elements of the set X such that $x < y$ and $y < z$. Let $T \cap R(\{x, y, z\}) = \{c\}$. Then from Lemma 12 it follows that $c(x) < c(y)$ and $c(y) < c(z)$. Since $\{c(x), c(y), c(z)\} = im(c) = M_3 = \{x_0, x_1, x_2\}$, we have $c(x) = x_0, c(y) = x_1, c(z) = x_2$. Finally, using Lemma 12 and $x_0 < x_2$, we get $x < z$. □

Lemma 14. *The element x_0 is the smallest element of the set X, that is, the inequality $x_0 < x$ for all $x \in X$ holds true.*

Proof. It is enough to consider the product $id_{\{x_0\}}a_{x_0,x}$. □

Lemma 15. *The relation $<$ is a well order, that is, every non-empty subset $Y \subseteq X$ has the smallest element.*

Proof. Let a be a unique element of the set $T \cap R(Y)$. From Lemma 10 it follows that $x_0 \in im(a)$. Let $y = a^{-1}(x_0)$. Then from Lemmas 12 and 14 it follows that y is the smallest element of the set Y. □

By α denote the order-type of the set $(X, <)$.

Lemma 16. *For all $A, B \in P$ such that $\xi(A) = \xi(B)$, we have $A = B$.*

Proof. Let $A, B \in P$ be the sets from the formulation. Then by Lemma 10 we have either $A \subseteq B$ or $B \subseteq A$. Without loss of generality we can assume that $A \subseteq B$. Consider an element a of T such that $im(a) = A$. Assume $A \neq B$, then there exists $z \in B \backslash A$. By g denote an isomorphism of the well-ordered sets B and A. Since g is bijective, we see that all elements of the sequence $\{g^{(n)}(z), n \geq 0\}$ are different and the set $C := \{g^{(n)}(z), n \geq 0\}$ is countable. Consider the pair $(z, g(z))$. If $z > g(z)$, then $g^{(n)}(z) > g^{(n+1)}(z)$ for all $n \geq 0$. This implies that the set C does not possess the smallest element. This contradicts Lemma 15. Thus $z < g(z)$ and z is the smallest element of C. Let b be a unique element of the set $T \cap R(C)$. Then $b(z) = x_0$. Since $ab \in T$, we see that there exists a unique number $k \geq 1$ such that $b(g^{(k)}(z)) = x_0$. This contradicts the injectivity of the map b and completes the proof of the lemma. □

Lemma 17. *For any ordinal number $\beta \leq \alpha$ the transformation $id_{X(\beta)}$ belongs to the cross-section T.*

Proof. The proof is by transfinitary induction on β. Since $0 \in T$, the basis of induction holds true. Assume the statement holds for all ordinal numbers less than β and denote by a a unique element of the set $T \cap R(X(\beta))$. Let us consider two cases.

1) β is nonlimiting ordinal. Then the set $X(\beta)$ has the greatest element $x_{\beta'}$ and $\beta = \beta' + 1$. By the inductive hypothesis, $id_{X(\beta')} \in T$. Now let $b = id_{X(\beta')}a$, then $dom(b) = X(\beta')$ and $b = id_{X(\beta')}$. This implies that $a|_{X(\beta')} = id_{X(\beta')}$. To complete the proof it is now enough to show that $a(x_{\beta'}) = x_{\beta'}$. Assume the converse. Then $a(x_{\beta'}) = x_\delta > x_{\beta'}$. Further, suppose c is a unique element of the set $T \cap R(x_\delta, x_{\beta'})$. Then since $rk(ac) = 1$, we obtain $c(x_\delta) = x_0$ and $c(x_{\beta'}) = x_1$. This contradicts Lemma 12 and so $a(x_{\beta'}) = x_{\beta'}$.

2) β is limiting ordinal. In this case, for all ordinal numbers γ such that $\gamma < \beta$, we have $\gamma + 1 < \beta$. Then by the inductive hypothesis, $id_{X(\gamma+1)} \in T$. In addition, let $b = id_{X(\gamma+1)}a$. Then $dom(b) = X(\gamma + 1)$ and $b = id_{X(\gamma+1)}$. This implies $a|_{X(\gamma+1)} = id_{X(\gamma+1)}$. In particular, $a(x_\gamma) = x_\gamma$. Therefore $a = id_{X(\beta)}$. This completes the proof of the lemma. □

By Lemma 12, Lemma 16 and Lemma 17, $T \subseteq R(X,<)$. But T and $R(X,<)$ contain a unique element from each $\mathcal{R}-$class of IS_X and so we must have $T = R(X,<)$.

Case 2. $|I| > 1$.

Lemma 18. *If $a \in T$, then $a(X_i) \subseteq X_i$ for all $i \in I$.*

Proof. Assume the converse. Then there exist elements $i \in I$ and $x \in X_i$ such that $a(x) \notin X_i$. Let b be a unique element of the set $T \cap R(\{a(x)\})$; then we obviously have $b(a(x)) = j \neq i$. Since $dom(ab) = \{x\}$ and $(ab)(x) = j$, we obtain $x \in X_j$ and so $x \notin X_i$. This contradiction completes the proof of the lemma. □

For any $i \in I$ consider the set $T_i = \{a \in T |\ dom(a) \subseteq X_i\}$ and denote $R_i := \{a|_{X_i} : a \in T_i\}$. Clearly, the set R_i is an $\mathcal{R}-$cross-section in IS_{X_i} and also it satisfies the condition of case 1. Hence $R_i = R(X_i, <_i)$ for some well order $<_i$ on the elements of X_i.

Lemma 19. *Let a be an arbitrary element of T and $x, y \in dom(a) \cap X_i$. If $x <_i y$, then $a(x) <_i a(y)$.*

Proof. The proof is analogous to one of Lemma 12. □

For any $i \in I$ by P_i denote the set $\{a(dom(a) \cap X_i)|\ a \in T\}$.

Lemma 20. *For any $i \in I$ and for all $A, B \in P_i$ we have either $A \subseteq B$ or $B \subseteq A$.*

Proof. The proof is analogous to one of Lemma 10. □

Lemma 21. *For any $i \in I$ and for all $A, B \in P$ such that $\xi(A) = \xi(B)$, we have $A = B$.*

Proof. The proof is analogous to one of Lemma 16. □

Lemma 22. *For all $a \in T$, we have $a(dom(a) \cap X_i) = X_i(\xi(dom(a) \cap X_i))$.*

Proof. Consider an element b of T such that $dom(b) = dom(a) \cap X_i$. From Lemma 19 it follows that $\xi(a(dom(a) \cap X_i)) = \xi(im(b))$. Also, since $b \in T_i$, we obtain $a(dom(a) \cap X_i) = im(b) = X_i(\xi(dom(a) \cap X_i))$ by Lemma 21. □

Now by Lemma 22, $T \subseteq R(\{\bigcup_{i \in I}(X_i, <_i)\})$. But both T and $R(\{\bigcup_{i \in I}(X_i, <_i)\})$ contain a unique element from each $\mathcal{R}-$class of IS_X and so we must have $T = R(\{\bigcup_{i \in I}(X_i, <_i)\})$. □

The anti-involution $a \mapsto a^{-1}$ interchanges $\mathcal{R}-$ and $\mathcal{L}-$classes in every inverse semigroup. Clearly, this anti-involution also maps $\mathcal{L}-$cross-sections to $\mathcal{R}-$cross-section and vice versa. Hence, dualizing Theorem 3, one immediately gets the description of the $\mathcal{L}-$cross-sections in IS_X. To formulate this theorem it is convenient to introduce the following notation.

Let α_i be the order-type of the set X_i. Now construct the set $L(\{\bigcup_{i\in I}(X_i,<_i)\})$ in the following way: an element $a \in L(A)$ belongs to $L(\{\bigcup_{i\in I}(X_i,<_i)\})$ if and only if the map $a|_{X_i(\eta_i)}$ is an isomorphism of $X_i(\eta_i)$ and $A\cap X_i$, where $\eta_i = \xi(A\cap X_i) \leq \alpha_i$ for all $i \in I$.

Theorem 4. *a) For an arbitrary decomposition $X = \bigcup_{i\in I} X_i$ and arbitrary well orders on the elements of every block of this decomposition the set $L(\{\bigcup_{i\in I}(X_i,<_i)\})$ is an $\mathcal{L}-$cross-section of IS_X.*

b) If $\{\bigcup_{i\in I}(X_i,<_i)\} \neq \{\bigcup_{j\in J}(X_j,<_j)\}$ then one has that $L(\{\bigcup_{i\in I}(X_i,<_i)\}) \neq L(\{\bigcup_{j\in J}(X_j,<_j)\})$.

c) Moreover, every $\mathcal{L}-$cross-section of IS_X has the form $L(\{\bigcup_{i\in I}(X_i,<_i)\})$ for some decomposition $X = \bigcup_{i\in I} X_i$ and some well orders $<_i$ on the elements of every block.

4. Classification of $\mathcal{R}-(\mathcal{L}-)$ cross-sections up to isomorphism

By $\omega_{\alpha+1}$ denote the smallest ordinal number of cardinality $\aleph_{\alpha+1}$. Let $R = R(\{\bigcup_{i\in I}(X_i,<_i)\})$ be an $\mathcal{R}-$cross-section of IS_X, where $|X| = \aleph_\alpha$. The map $f_R : W(\omega_{\alpha+1}) \to [0,\aleph_\alpha], \eta \mapsto |\{i\in I|\xi(X_i)=\eta\}|$, will be called the *type* of R. Analogously one defines the type of an $\mathcal{L}-$cross-section.

Theorem 5. *Two $\mathcal{R}-(\mathcal{L}-)$ cross-sections in IS_X are isomorphic if and only if they have the same type.*

Proof. Clearly, it is enough to prove the statement for, say $\mathcal{R}-$cross-sections. Let $R_1 = R(\{\bigcup_{i\in I}(X_i,<_i)\})$ and $R_2 = R(\{\bigcup_{j\in J}(X_j,<_j)\})$ be two arbitrary $\mathcal{R}-$cross-sections of types f_{R_1} and f_{R_2} respectively.

Necessity. Assume first that $R_1 \simeq R_2$ and f is an arbitrary isomorphism of these cross-sections. Since every idempotent of the cross-sections has the form id_A for some subset $A \subseteq X$, we have $f(id_A) = id_B$. Consider the equation $id_A \cdot x = x$ in the semigroup R_1. Its solutions form the set $\{x \in R_1|\ dom(x) \in A\}$. Since R_1 is a cross-section, this equation has exactly $2^{|A|}$ solutions. Also, since corresponding equations have the same quantity of solutions under the isomorphism, we obtain $2^{|A|} = 2^{|B|}$. Thus if $|A| = n < \infty$, then $|B| = n$. This implies that there exists a bijection between idempotents of the finite rank n. In particular, for

$n = 1$, there exists a bijection $\widetilde{f}$ between the set I and the set J given by the rule $\widetilde{f}(i) = j$ iff $f(id_{\{i\}}) = id_{\{j\}}$. For any $i \in I$ and $x \in X_i$ by a_x denote a unique element of $R_1 \cap R(\{x\})$. Further, for any $i \in I$ define the map $f_i : X_i \to Y_{\widetilde{f}(i)}$ by the rule $x \mapsto dom(f(a_x))$. Since $f(a_x)$ satisfies the equation $y \cdot id_{\{\widetilde{f}(i)\}} = y$, we see that this map is well defined. Also, since f is isomorphism, we see that f_i is bijection. To complete the proof it is now enough to show that f_i is an isomorphism of the well-ordered sets X_i and $Y_{\widetilde{f}(i)}$ for all $i \in I$. Since for all $a \in R_1$ such that $rk(a) = n < \infty$, we have $a \cdot id_{im(a)} = a$, where $rk(id_{im(a)}) = n$, we obtain that in the semigroup R_2 the equality $f(a) \cdot id_B = f(a)$ holds true, where $rk(id_B) = n$. This implies $rk(f(a)) \leq n = rk(a)$. Similarly, we can show that $rk(a) \leq rk(f(a))$ and so for all $a \in R_1$ such that $rk(a) = n < \infty$, we have $rk(a) = rk(f(a))$.

Let an element i of the set I be fixed. If $|X_i| = 1$, then it is obvious, that f_i is an isomorphism of the well-ordered sets. If $|X_i| > 1$, then by i' denote the successor of i in the well-ordered set $(X_i, <_i)$. Let $j := f_i(i) = \widetilde{f}(i), j' := f_i(i')$, $a_i = id_{\{i\}}$ and $b_j = id_{\{j\}}$. Suppose $a_{i'}$ is a unique element of $R_1 \cap R(\{i'\})$ and $b_{j'}$ is a unique element of $R_2 \cap R(\{j'\})$. Then $f(a_i) = b_j$ and $f(a_{i'}) = b_{j'}$. Since $id_{\{i,i'\}} \in R_1$, we see that in R_1 the equalities $id_{\{i,i'\}} \cdot a_i = a_i, id_{\{i,i'\}} \cdot a_{i'} = a_{i'}$ hold true. Therefore in R_2 the equalities $f(id_{\{i,i'\}}) \cdot b_j = b_j, f(id_{\{i,i'\}}) \cdot b_{j'} = b_{j'}$ hold true. This implies $j, j' \in dom(f(id_{\{i,i'\}}))$ and $f(id_{\{i,i'\}})|_{\{j,j'\}} = id_{\{j,j'\}}$. But since $rk(f(id_{\{i,i'\}})) = 2$, we obtain $f(id_{\{i,i'\}}) = id_{\{j,j'\}}$. Hence $id_{\{j,j'\}} \in R_2$. This means that j' is the successor of j in the well-ordered set $(Y_j, <_j)$.

Let x_1 and x_2 be two different elements of X_i such that $x_1 <_i x_2$. Then the element $\alpha = \begin{pmatrix} x_1 & x_2 \\ i & i' \end{pmatrix}$ belongs to R_1. Let $y_1 := f_i(x_1), y_2 := f_i(x_2)$. Defining elements $a_{x_1}, a_{x_2}, b_{y_1}, b_{y_2}$ similarly, we can show that $\begin{pmatrix} y_1 & y_2 \\ j & j' \end{pmatrix} = f(\alpha) \in R_2$. This means $y_1 <_j y_2$. Therefore f_i is an isomorphism of the well-ordered sets $(X_i, <_i)$ and $(Y_j, <_j)$ and the statement is proved.

Sufficiency. Obvious. □

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