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Realizations of affine Lie algebras

Vyacheslav Futorny

Dedicated to Yu.A. Drozd on the occasion of his 60th birthday

Abstract. This is an expository paper on some realizations of affine Lie algebras and their representations in terms of differential operators. In particular we recall the classical construction of Wakimoto modules and its recent generalization - Intermediate Wakimoto modules [CF].

1. Introduction

These notes are based on my talk at the conference in honor of Yuriy Drozd at Kiev University in December of 2004.

Let $\mathfrak{G}$ be a complex simple finite-dimensional Lie algebra, $\mathfrak{G} = Lie\ G$. The classification of such algebras goes back to Killing and Cartan. These algebras are parameterized by Cartan matrices (a_{ij}) with non-positive integer entries, which are positive definite and satisfy the conditions: $a_{ii} = 2$, $a_{ij} = 0 \Rightarrow a_{ji} = 0$ for all i, j. In 1967 Kac and Moody generalized this to a new class of Lie algebras, known today as Kac-Moody algebras, by relaxing the condition of Cartan matrix to be positive definite. A particular case of positive semidefinite matrix ($\det(a_{ij}) = 0$, with positive principal minors) corresponds to *affine* Lie algebras. We address to [K] for the basics of the Kac-Moody theory.

Kac-Moody algebras have wide applications in quantum field theory, combinatorics, knot theory, group theory, number theory and harmonic analysis among others. Affine Lie algebras are the most studied among infinite-dimensional Kac-Moody algebras and with most of the applications. One of the reasons of their popularity is the existence of concrete realizations of these algebras.

Let $\hat{\mathfrak{G}} = \mathfrak{G} \otimes \mathbb{C}[t, t^{-1}]$ with the Lie bracket

$$[x \otimes t^n, y \otimes t^m] = [x, y] \otimes t^{m+n},$$

for all $x, y \in \mathfrak{G}$, $n, m \in \mathbb{Z}$. This is so-called *loop* algebra associated with $\mathfrak{G}$. This is the Lie algebra of the group of maps $Map(S^1, G)$ of the circle S^1 to the Lie group G. Loop algebras have a 1-dimensional universal central extension $\mathfrak{g} = \hat{\mathfrak{G}} \oplus \mathbb{C}c$,

$$[x \otimes t^n, c] = 0, [x \otimes t^n, y \otimes t^m] = [x, y] \otimes t^{m+n} + n < x, y > \delta_{n+m,0}c.$$

Finally, let $d : \mathfrak{g} \to \mathfrak{g}$ be a degree derivation, $d(x \otimes t^n) = n(x \otimes t^n)$, $d(c) = 0$, for all $x \in \mathfrak{G}$, $n \in \mathbb{Z}$. The algebra

$$\tilde{\mathfrak{G}} = \mathfrak{g} \oplus \mathbb{C}d,$$

where $[d, x] = d(x)$ for all $x \in \mathfrak{g}$, is a *non-twisted affine* Kac-Moody algebra. Here $<,>$ denotes the Killing form on $\mathfrak{G}$. Twisted affine Lie algebras correspond to fixed points of diagram automorphisms. In these notes we will only deal with the non-twisted case. The realization above leads to many applications of the affine Lie algebras in the theory of theta functions, modular forms and soliton theory ([23]-[25]).

Let $U(\mathfrak{G})$ be the universal enveloping algebra of $\mathfrak{G}$, $Z(\mathfrak{G})$ the center of $U(\mathfrak{G})$. Let $\mathcal{H}$ be a Cartan subalgebra of $\mathfrak{G}$, $\dim \mathcal{H} = rank\mathfrak{G}$. Then $\tilde{\mathcal{H}} = \mathcal{H} \oplus \mathbb{C}c \oplus \mathbb{C}d$ is a Cartan subalgebra of $\tilde{\mathfrak{G}}$. We will denote by A_n the n-th Weyl algebra of differential operators acting in the space of polynomials $\mathbb{C}[x_1, \ldots, x_n]$, generated by $x_1, \ldots, x_n$ and partial derivatives $\partial_1, \ldots, \partial_n$. If $\xi : \mathbb{Z}(\mathfrak{G}) \to \mathbb{C}$ is the central character of $\mathfrak{G}$ then the quotient $U(\mathfrak{G})/(Ker\xi)U(\mathfrak{G})$ can be embedded into A_n, $n = (1/2)(\dim \mathfrak{G} - rank\mathfrak{G})$ [Co], providing a realization of $\mathfrak{G}$ in the Fock space $\mathbb{C}[x_1, \ldots, x_n]$. A different approach was suggested by Khomenko [Kh], who showed that the quotient $U(gl(n))/(Ker\xi)U(gl(n))$ can be embedded into a certain localization of A_m, $m = n(n+1)/2$, using the theory of Gelfand-Tsetlin modules [DFO]. Such realizations are extremely useful for the representation theory of these algebras. The generators of the Weyl algebra A_n are called *bosons* in physics literature. Hence embeddings above can be viewed as boson type realizations of the Lie algebra $\mathfrak{G}$.

The goal of these notes is to discuss various realizations of affine Lie algebras via differential operators on some Fock spaces, in particular, new boson type realizations recently obtained by B.Cox and the author [CF].

2. Verma type modules

Let $\mathfrak{a}$ be a Lie algebra with a Cartan subalgebra H and root system Δ. A closed subset $P \subset \Delta$ is called a partition if $P \cap (-P) = \emptyset$ and $P \cup (-P) = \Delta$. If $\mathfrak{a}$ is finite-dimensional then every partition corresponds to a choice of positive roots in Δ and all partitions are conjugate by the

Weyl group. The situation is different in the infinite-dimensional case. If $\mathfrak{a}$ is an affine Lie algebra then partitions are divided into a finite number of Weyl group orbits (cf. [JK], [F2]).

Given a partition P of Δ we define a *Borel* subalgebra $\mathfrak{b}_P \subset \mathfrak{a}$ generated by H and the root spaces $\mathfrak{a}_\alpha$ with $\alpha \in P$. All Borel subalgebras are conjugate in the finite-dimensional case. A *parabolic* subalgebra is a subalgebra that contains a Borel subalgebra. If $\mathfrak{p}$ is a parabolic subalgebra of a finite-dimensional $\mathfrak{a}$ then $\mathfrak{p} = \mathfrak{p}_0 \oplus \mathfrak{p}_+$ where $\mathfrak{p}_0$ is a reductive Levi factor and $\mathfrak{p}_+$ is a nilpotent subalgebra. Parabolic subalgebras correspond to a choice of a basis π of the root system Δ and a subset $S \subset \pi$. A classification of all Borel subalgebras in the affine case was obtained in [F2]. In this case not all of them are conjugate but there exists a finite number of conjugacy classes. These conjugacy classes are parametrized by parabolic subalgebras of the underlined finite-dimensional Lie algebra. Namely, let $\mathfrak{p} = \mathfrak{p}_0 \oplus \mathfrak{p}_+$ a parabolic subalgebra of $\mathfrak{G}$ containing a fixed Borel subalgebra $\mathfrak{b}$ of $\mathfrak{G}$. Define

$$B_{\mathfrak{p}} = \mathfrak{p}_+ \otimes \mathbb{C}[t, t^{-1}] \oplus \mathfrak{p}_0 \otimes t\mathbb{C}[t] \oplus \mathfrak{b} \oplus \mathbb{C}c \oplus \mathbb{C}d.$$

For any Borel subalgebra $\mathfrak{B}$ of $\tilde{\mathfrak{G}}$ there exists a parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{G}$ such that $\mathfrak{B}$ is conjugate to $B_{\mathfrak{p}}$.

When $\mathfrak{p}$ coincides with $\mathfrak{G}$, i.e. $\mathfrak{p}_+ = 0$, the corresponding Borel subalgebra $B_{\mathfrak{G}}$ is the *standard* Borel subalgebra defined by the choice of positive roots in $\tilde{\mathfrak{G}}$. Another extreme case is when $\mathfrak{p}_0 = \mathcal{H}$. This corresponds to the *natural* Borel subalgebra B_{nat} of $\tilde{\mathfrak{G}}$ considered in [JK].

Given a parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{G}$ let $\lambda : B_{\mathfrak{p}} \to \mathbb{C}$ be a 1-dimensional representation of $B_{\mathfrak{p}}$. Then one defines an induced *Verma type* $\tilde{\mathfrak{G}}$-module

$$M_{\mathfrak{p}}(\lambda) = U(\tilde{\mathfrak{G}}) \otimes_{U(B_{\mathfrak{p}})} \mathbb{C}.$$

The module $M_{\mathfrak{G}}(\lambda)$ is the classical Verma module with highest weight λ [K]. In the case of natural Borel subalgebra we obtain *imaginary* Verma modules studied in [F1]. Note that the module $M_{\mathfrak{p}}(\lambda)$ is $U(\mathfrak{p}_-)$-free, where $\mathfrak{p}_-$ is the opposite subalgebra to $\mathfrak{p}_+$. The theory of Verma type modules was developed in [F2]. It follows immediately from the definition that, unless it is a classical Verma module, Verma type module with highest weight λ has a unique maximal submodule, it has both finite and infinite-dimensional weight spaces and it can be obtained using the parabolic induction from a standard Verma module M with highest weight λ over a certain affine Lie subalgebra. Moreover, if the central element c acts non-trivially on such Verma type module then the structure of this module is completely determined by the structure of module M, which is well-known ([F2], [C1]).

Let V be a weight $\mathfrak{a}$-module, i.e. $V = \oplus_{\mu \in H^*} V_\mu$, $V_\mu = \{v \in V | hv = \mu(h)v, \forall h \in H\}$. Suppose that $\dim V_\mu \leq \infty$ for all μ. If $\mathfrak{a}$ is a Kac-Moody Lie algebra (finite or affine) with Serre generators e_i's and f_i's, then denote by w an anti-involution on $\mathfrak{a}$ which permutes e_i with f_i for all i, and which is identity on H. Consider a $\mathfrak{a}$-module

$$V^* = \oplus_{\mu \in H^*} V_\mu^*,$$

where V_μ^* is a dual subspace of V_μ and the structure of $\mathfrak{a}$-module is given by: $(xf)(v) = f(w(x)v)$. Such modules are called *contragradient*. The advantage of considering these modules versus the whole dual modules of V is that V and its contragradient module belong to the Bernstein-Gelfand-Gelfand category $\mathcal{O}$ simultaneously.

3. Flag varieties

Let $G = GL(n, \mathbb{C})$ and B the subgroup of upper triangular matrices in G.

Let V be an n-dimensional complex vector space and $e_1, \ldots, e_n$ a basis of V. The group G acts naturally on V as a matrix multiplication on n-tuples of the coordinates with respect to the basis $e_1, \ldots e_n$. A full *flag* of subspaces is a chain

$$0 \subset V_1 \subset \ldots \subset V_n = V$$

of subspaces of V of length n. Denote by Ω the set of all full flags in V. Then the group G acts transitively on Ω. Denote $V^i = span\{e_1, \ldots e_i\}$, $i = 1, \ldots, n$. These subspaces form a canonical full flag

$$0 \subset V^1 \subset \ldots \subset V^n = V.$$

in V and B is its stabilizer. Hence Ω can be identified with G/B, which is a homogeneous space for G.

Let now $\mathfrak{G}$ be a finite-dimensional simple Lie algebra, $\mathfrak{b}$ a Borel subalgebra, $\mathfrak{G} = LieG$, $\mathfrak{b} = LieB$. Then B is a maximal solvable subgroup in G which is called *Borel subgroup*. All Borel subgroups are conjugate. Hence the group G acts transitively on the *flag variety* $X = G/B$, which is a smooth algebraic variety and can be identified with the set of all Borel subalgebras of $\mathfrak{G}$. It can also be viewed as a complex manifold.

If $x \in X$ then the differential of the map $g \mapsto gx$, $g \in G$, defines a map of $\mathfrak{G}$ onto the tangent space of X at point x. Hence we have an action of the Lie algebra $\mathfrak{G}$ on X. Let $\mathcal{O}_X$ denote the structure sheaf on X and let $\mathcal{D}_X$ be the sheaf of differential operators on X with regular coefficients. If $\mathcal{M}$ is a $\mathcal{D}_X$-module then the global sections $\Gamma(X, \mathcal{M})$ have a structure of a $\mathfrak{G}$-module. In particular, if λ is an antidominant weight and $\mathcal{O}_X(\lambda)$ is the G-equivariant line bundle, corresponding to λ, then the global sections $\Gamma(X, \mathcal{O}_X(\lambda))$ is an irreducible finite-dimensional $\mathfrak{G}$-module with lowest weight λ by the celebrated Borel-Weil theorem.

If $\lambda \in \mathcal{H}^*$ then the Verma module $M(\lambda)$ with the highest weight λ admits the central character which we denote by ξ_λ. Let $\xi : Z(\mathfrak{G}) \to \mathbb{C}$ and $\lambda \in \mathcal{H}^*$ is such that $\xi = \xi_\lambda$. Consider $U_\lambda = U(\mathfrak{G})/U(\mathfrak{G})Ker\xi$. Beilinson and Bernstein ([BB]) introduced a *twisted sheaf* of differential operators $\mathcal{D}_\lambda$ on X and showed that there is an isomorphism of algebras

$$U_\lambda \simeq \Gamma(X, \mathcal{D}_\lambda).$$

Moreover, there is an equivalence between the category of U_λ-modules and the category of quasi-coherent $\mathcal{D}_\lambda$-modules on X [BB]. This equivalence is given by the functors of localization and global sections.

4. Boson-fermion correspondence

Affine Lie algebras proved to be very useful for the theory of soliton equations. In particular, the affine Lie algebra $\hat{sl(2)}$, corresponding to $sl(2)$, is related to the solutions of Korteweg-de Vries (KdV) hierarchies [DS].

The KdV equation

$$u_t = 6uu_x - u_{xxx}$$

appeared in 1895 and described the behavior of an isolated wave in a narrow canal, which was first observed by Rassell in 1834.

Application of the representation theory of $\hat{sl(2)}$ to construct soliton type solutions of the KdV hierarchies is based on the vertex operator realization (*principal realization*) of the basic highest weight module [LW]. In this realization of $\hat{sl(2)}$ the Fock space is the polynomial ring in infinitely many odd variables with positive indices. An amazing fact is that there exists a realization of the affine Lie algebra $\hat{sl(2)}$ for which the Fock space is the polynomial ring in infinitely many odd variables, with both positive and negative indices. In fact, Billig ([B]) showed that $\hat{sl(2)}$ acts on the Weyl algebra with infinitely many odd generators, and that this action is related to the soliton type solutions of the *sine-Gordon* equation

$$u_{xt} = \sin(u).$$

This equation describes the surfaces of constant negative curvature in $\mathbb{R}$.

Realization mentioned above is based on the boson-fermion correspondence, which is a way to construct free fermions (Clifford algebra) out of free bosons (Weyl algebra), using the physics terminology. Free fermions ψ_i, ψ_i^*, $i \in \mathbb{Z}$, generate the Clifford algebra $\mathcal{C}$ with the relations

$$\psi_i\psi_j^* + \psi_j^*\psi_i = \delta_{ij},\ \psi_i\psi_j + \psi_j\psi_i = 0,\ \psi_i^*\psi_j^* + \psi_j^*\psi_i^* = 0,$$

$i, j \in \mathbb{Z}$. Consider the generating series

$$\psi(z) = \sum_{k\in\mathbb{Z}} \psi_k z^{-k},\ \psi^*(z) = \sum_{k\in\mathbb{Z}} \psi_k^* z^k.$$

Let $\mathcal{A}$ be the Weyl algebra with generators p_i, q_i, $i \in \mathbb{Z}$ subject to the relations:

$$[p_i, p_j] = [q_i, q_j] = 0,\ [p_i, q_j] = -4/j\delta_{ij}.$$

It was shown in [B] (cf. [K], Theorem 14.10) that the Clifford algebra has a natural representation

$$\psi(z) \mapsto [z]u \exp(1/2 \sum_{j>0} q_j z^j) \exp(1/2 \sum_{j>0} p_j z^{-j}),$$

$$\psi^*(z) \mapsto u^{-1}[z]^{-1} \exp(-1/2 \sum_{j>0} q_j z^j) \exp(-1/2 \sum_{j>0} p_j z^{-j})$$

in the space $\mathbb{C}[u, u^{-1}] \otimes \bar{\mathcal{A}}$, where $\bar{\mathcal{A}}$ is a certain completion of $\mathcal{A}$ and $[a]u^m = a^m u^m$.

On the other hand, Feingold and Frenkel ([FdF]) showed that any classical affine Lie algebra can be embedded into the Clifford algebra $\mathcal{C}$. Hence classical affine Kac-Moody algebras can be realized in operators acting on differential operators by left multiplication.

Example 4.1. *Consider the case of the affine Lie algebra* $\mathfrak{g} = sl(2) \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c$. *It is well-known that the following elements span the loop subalgebra of* $\mathfrak{g}$: $H_{2j+1} = e \otimes t^j + f \otimes t^{j+1}$, $a_{2j} = -h \otimes t^j$, $a_{2j+1} = e \otimes j - f \otimes t^{j+1}$. *Let* $b_{ij} = \psi_i\psi_j^*$ *if* $i \neq j$ *or* $i = j > 0$ *and* $b_{ii} = -\psi_i^*\psi_i$ *if* $i \leq 0$. *The correspondence*

$$H_{2j+1} \mapsto \sum_{i\in\mathbb{Z}} b_{i,i+2j+1},\ a_j \mapsto \sum_{i\in\mathbb{Z}} (-1)^{i+j} b_{i,i+j},\ c \mapsto 1,$$

$j \in \mathbb{Z}$, *together with the representation of the Clifford algebra described above, yields:*

$$H_m \mapsto (-m/2)p_m,\ H_{-m} = (m/2)q_m,\ m \in \mathbb{N}_{odd},$$

$$\sum_{i\in\mathbb{Z}} a_i z^{-i} \mapsto 1/2([-1]\Gamma(z) - 1),$$

where

$$\Gamma(z) = \exp(\sum_{j\in\mathbb{N}_{odd}} q_j z^j) \exp(\sum_{j\in\mathbb{N}_{odd}} p_j z^{-j})$$

is a vertex operator with the origin in the string theory.

5. Boson type realizations of Verma modules

5.1. Finite-dimensional case

Consider first a finite-dimensional case. Let $\mathfrak{G} = LieG$ be a simple finite-dimensional Lie algebra with a Cartan decomposition $\mathfrak{G} = \mathfrak{n}_- \oplus \mathcal{H} \oplus \mathfrak{n}_+$. Take a Borel subalgebra $\mathfrak{b} = \mathfrak{n}_- \oplus \mathcal{H}$. Let $\mathfrak{b} = LieB$, $\mathfrak{n}_\pm = LieN_\pm$. Consider the flag variety $X = G/B$. Then X has a decomposition into open Schubert cells: $X = \cup_{w \in W} C(w)$, where $C(w) = B_+ w B_-/B_-$, $W = N(T)/T$ is the Weyl group and $T = B_+/N_+$. The codimension of $C(w)$ equals the length of w. The subgroup N_+ acts on X, and the largest orbit $\mathcal{U}$ of this action can be identified with proper N_+. From Section 3 we know that the Lie algebra $\mathfrak{G}$ can be mapped into vector fields on X and hence on $\mathcal{U}$. Thus $\mathfrak{G}$ can be embedded into the differential operators on $\mathcal{U}$ of degree ≤ 1. Since $N_+ \simeq \mathfrak{n}_+$ via the exponential map, we conclude that N_+ can be identified with an affine space $\mathbb{A}^{|\Delta_+|}$. Therefore, the ring of regular functions $\mathcal{O}_\mathcal{U}$ on $\mathcal{U}$ is just a polynomial ring in $m = |\Delta_+|$ variables and $\mathfrak{G}$ has an embedding into the Weyl algebra $\mathcal{A}_m$. In particular, $\mathcal{O}_\mathcal{U}$ is a $\mathfrak{G}$-module. In fact, a $\mathfrak{G}$-module $\mathcal{O}_\mathcal{U}$ is isomorphic to a contragradient module $M^*(0)$ with trivial highest weight.

Example 5.1. *Let $\mathfrak{G} = sl(2)$ with a standard basis e, f, h, $[e, f] = h$, $[h, e] = 2e$, $[h, f] = -2f$. Let $\mathfrak{b}_- = span\{f, h\}$. Then $G = SL_2(\mathbb{C})$ and the variety $X = G/B_-$ can be identified with the projective line $\mathbb{P}^1$ which has a big cell $\mathcal{U} = \mathbb{A}^1$. Let y be a coordinate in $\mathcal{U}$. The group G acts on X in such a way that an element*

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

changes y into $(ay + b)/(cy + d)$.

Denote $\mathcal{O}_\mathcal{U} = \mathbb{C}[x]$. Then one computes

$$e \mapsto d/dx, h \mapsto -2xd/dx, f \mapsto -x^2 d/dx.$$

Hence, $M^*(0) \simeq \mathbb{C}[x]$ as a $\mathfrak{G}$-module (for details see Example 10.2.1 in [FZ]).

For a general $\lambda \in \mathcal{H}^*$, $\Gamma(\mathcal{U}, \mathcal{D}_\lambda)$ is a $\mathfrak{G}$-module under the inclusion

$$\mathfrak{G} \subset \Gamma(G/B_-, \mathcal{D}_\lambda),$$

and this module is isomorphic to $M^*(\lambda)$ (cf. Remark 10.2.7 in [FZ]). In the case of an integral highest weight λ, the contragradient module

$M^*(\lambda)$ can be realized as a zero cohomology group $H^0(X, \mathcal{L}_\lambda)$ of the line bundle $\mathcal{L}_\lambda$ on the flag manifold with the support in the big cell.

In order to obtain a geometrical realization of Verma modules one needs to consider the minimal 1-point orbit of N_+ on X. In the example above any matrix

$$\begin{pmatrix} a & b \\ c & 0 \end{pmatrix},$$

where $bc = -1$ serves as a representative of this orbit.

Choosing another orbit of N_+ will give us a *twisted* Verma module. Twisted Verma modules are parametrized by the elements of the Weyl group and have the same character as corresponding Verma modules. For integral λ these modules can be realized as local cohomology groups $H^{l(w)}_{\mathcal{C}(w)}(X, \mathcal{L}_{w^{-1}\cdot\lambda})$ of the line bundle $\mathcal{L}_{w^{-1}\cdot\lambda}$ with the support in the Bruhat cell $\mathcal{C}(w)$. For the detailed account on twisted Verma modules see [AL].

Remark 5.2. *Consider again example of $sl(2)$. Another way to get a realization of Verma module $M(0)$ on Fock space $\mathbb{C}[x]$ is the following. Apply to $sl(2)$ an automorphism which is a composition of two anti-involutions: $e \leftrightarrow f$, h is fixed, and $x \leftrightarrow d/dx$. Then it gives the following realization in second order differential operators: $f \mapsto x$, $h \mapsto -2xd/dx$, $e \mapsto -x(d/dx)^2$.*

This approach will be particularly useful in the affine case.

5.2. Affine case

Consider now the loop algebra $\hat{\mathfrak{G}} = \mathfrak{G} \otimes \mathbb{C}[t, t^{-1}]$. We follow closely [FZ] in this section. Sometimes it is more convenient to consider a completion of $\hat{\mathfrak{G}}$ substituting the Laurent polynomials by the Laurent power series. We will denote this Lie algebra by $\mathfrak{G}((t))$. The corresponding loop group will be denoted by $G((t))$. Fix a Cartan decomposition $\mathfrak{G} = \mathfrak{n}_- \oplus \mathcal{H} \oplus \mathfrak{n}_+$ and consider a Borel subalgebra $\mathfrak{b}_\pm = \mathfrak{n}_\pm \oplus \mathcal{H}$. Denote

$$\hat{\mathfrak{n}}_\pm = (\mathfrak{n}_\pm \otimes 1) \oplus (\mathfrak{G} \otimes t^\pm \mathbb{C}[[t^\pm]]),$$

$\hat{\mathfrak{b}}_\pm = \hat{\mathfrak{n}}_\pm \oplus \mathcal{H} \otimes \mathbb{C}[[t]]$. Let $\hat{N}_\pm$ and $\hat{B}_\pm$ be Lie groups corresponding to $\hat{\mathfrak{n}}_\pm$ and $\hat{\mathfrak{b}}_\pm$ respectively. Consider a flag variety $X = G((t))/\hat{B}_-$ which has a structure of a scheme of infinite type. As in the finite-dimensional case X splits into $\hat{N}_+$-orbits of finite codimension, parametrized by the affine Weyl group. There is an analogue of a big cell $\hat{\mathcal{U}}$ in X which is a projective limit of affine spaces, and hence, the ring of regular functions $\mathcal{O}_\mathcal{U}$ on $\mathcal{U}$ is a polynomial ring in infinitely many variables. Thus $\mathfrak{G}((t))$ acts on it by differential operators providing a realization for the contragradient Verma module with zero highest weight. Global sections of more general $\hat{N}_+$-equivariant sheaves on X will produce an arbitrary highest weight. Other $\hat{N}_+$-orbits in X correspond to twisted contragradient Verma modules. A striking difference with the finite-dimensional case is that we can not obtain standard Verma modules this way. They can be obtained considering $\hat{N}_+$-orbits on $G((t))/\hat{B}_+$.

6. First free field realization

In the previous section we considered the case of classical Verma modules for affine Lie algebras. Consider the completion $\mathfrak{b}_{nat}$ in $\mathfrak{G}((t))$ of the natural Borel subalgebra $\mathfrak{n}_-\otimes\mathbb{C}[t,t^{-1}]\oplus\mathcal{H}\otimes\mathbb{C}[t^{-1}]$. If N_- is the Lie group corresponding to $\mathfrak{n}_-$ then $\mathfrak{B}_{nat} = N_-((t))\mathcal{H}[[t^{-1}]]$ is the Borel subgroup corresponding to $\mathfrak{b}_{nat}$. Let $X = G((t))/\mathfrak{B}_{nat}$. The difference with the classical case is that X is not a scheme. This structure is called the *semi-infinite* manifold [FZ], [V1]. It can be viewed as the space of maps from $Spec\mathbb{C}((t))$ to the finite-dimensional flag variety G/B_-. We can consider the $\hat{N}_+$-orbits on the semi-infinite manifold and, in particular, $N_+((t))$ can be viewed as an analogue of the big cell $\mathcal{U}$ in G/B_-.

Example 6.1 ([FZ], 10.3.6)**.** *For simplicity we will only consider the case $\mathfrak{G} = sl(2)$. The corresponding semi-infinite manifold can be thought as $\mathbb{P}^1((t))$. The big cell $\mathbb{A}^1 \subset \mathbb{P}^1$ can be lifted to a big cell $\mathbb{A}^1((t)) = \{\begin{pmatrix} x(t) & -1 \end{pmatrix}^t\}$, which coincides with the space of functions $F \simeq \mathbb{C}((t))$ on the punctured disc with the chosen coordinate t on the disc (though all constructions are coordinate free). Here $x(t) = \sum_{m\in\mathbb{Z}} x_m t^m \in \mathbb{C}((t))$. Denote by $\mathcal{F}$ the space of functions on F which consists of the series of type $g_0 + \sum_{n<0} g_n x_n$, where $g_0 \in \mathbb{C}[x_m, m \in \mathbb{Z}_+]$, $g_n \in \mathbb{C}[x_m, m \in \mathbb{Z}]$. The Lie algebra $sl(2)((t))$ acts naturally on F by vector fields. These vector fields can be described as series $\sum_{m\in Z} g_m \partial x_m$, where $g_m \in \mathcal{F}$ with the property that for any $M > 0$ there exists $K \leq M$ such that for each $m \leq K$, g_m is generated by x_k, $k \leq M$. Denote*

$$e_n = e\otimes t^n,\ h_n = h\otimes t^n,\ f_n = f\otimes t^n, n\in\mathbb{Z}.$$

Then the corresponding representation by vector fields on $\mathcal{F}$ is the following

$$e_n \mapsto \partial x_n,\ \ h_n \mapsto -2\sum_{m\in\mathbb{Z}} x_m \partial x_{n+m},\ \ f_n \mapsto -\sum_{m,k\in\mathbb{Z}} x_m x_k \partial x_{n+m+k}.$$

Consider the space $V = \mathbb{C}[x_m, m\in\mathbb{Z}]$. We would like to interpret V as a Fock space for $\hat{sl}(2)$ using the realization via vector fields above.

But it is clear that the differential operators corresponding to f_n are not well-defined on V (they take values in some formal completion of V). One way to deal with this problem is to apply the anti-involutions:

$$e_n \leftrightarrow f_n, \quad h_n \leftrightarrow h_n; \quad x_n \leftrightarrow \partial x_n, \ n \in \mathbb{Z}$$

which gives the following formulas:

$$f_n \mapsto \partial x_n, \ \ h_n \mapsto -2\sum_{m\in\mathbb{Z}} x_{n+m}\partial x_m, \ \ e_n \mapsto -\sum_{m,k\in\mathbb{Z}} x_{n+m+k}\partial x_m \partial x_k.$$

These formulas define the *first free field realization* of $\hat{sl}(2)$ in the polynomial ring $\mathbb{C}[x_m, m \in \mathbb{Z}]$. This module is, in fact, a quotient $M(0)$ of the imaginary Verma module with trivial highest weight by a submodule generated by the elements $h_n \otimes 1$, $n < 0$. Similar formulas for an arbitrary highest weight with a trivial action of the central element were obtained by Jakobsen and Kac [JK] using analytic approach. Namely, if μ denotes a finite measure on the circle S^1, not concentrated in a finite number of points, and $\lambda_m = \int_{S^1} z^m d\mu$ then

$$f_n \mapsto \partial x_n, \quad h_n \mapsto -\lambda_n - 2\sum_{m\in\mathbb{Z}} x_{n+m}\partial x_m,$$

$$e_n \mapsto -\sum_{m,k\in\mathbb{Z}} x_{n+m+k}\partial x_m \partial x_k - \sum_{m\in\mathbb{Z}} \lambda_{n+m}\partial x_m$$

gives a boson type realization of $M(\lambda)$, where $\lambda(c) = 0$ and $\lambda(h_0) = -\lambda_0$. This module is irreducible if $\lambda_0 \neq 0$ [F1]. This is the case in particular when μ is the Lebesgue measure on S^1 and $\lambda_m = \delta_{m,0}$.

To get a boson type realization of the imaginary Verma module for $\hat{sl}(2)$ with a non-trivial central action Bernard and Felder used the Borel-Weil construction. Let $\hat{B}_-$ be the Borel subgroup of the loop group $\hat{SL}(2)$ corresponding to a Borel subalgebra $\mathfrak{b}_{nat}$. Then $\hat{B}_-$ consists of the elements

$$\exp(\sum_{n\in\mathbb{Z}} x_n e_n) \exp(\sum_{m>0} y_m h_m),$$

where x_n, y_m are coordinate functions. Consider a one dimensional representation $\chi : \hat{B}_- \to \mathbb{C}$, where c acts by scalar K, h_0 acts by scalar J ($J/2$ is called the *spin*) and all other elements act trivially. Then one can construct a line bundle over $\hat{SL}(2)/\hat{B}_-$ by taking a fiber product

$$\mathcal{L}_\chi = \hat{SL}(2) \times_{\hat{B}_-} \mathbb{C}$$

and a map $g : \hat{SL}(2) \times_{\hat{B}_-} \mathbb{C} \to \hat{SL}(2)/\hat{B}_-$ such that $(x, z) \mapsto x\hat{B}_-$. The group $\hat{SL}(2)$ acts on the sections of the line bundle (i.e. the functions on $\hat{SL}(2)$ with certain conditions) by

$$(g_1 f)(g_2) = f(g_1^{-1} g_2),$$

$g_i \in \hat{SL}(2)$. Differentiating this action to an action of the Lie algebra $\mathfrak{g}$ and applying two anti-involutions

$$e_n \leftrightarrow -f_{-n},\ h_n \leftrightarrow h_{-n},\ \ c \leftrightarrow c$$

and

$$x_{-n} \leftrightarrow \partial x_n, \quad y_k \leftrightarrow -\partial y_k.$$

we obtain the following boson realization of $\mathfrak{g}$ in the Fock space $\mathbb{C}[x_m, m \in \mathbb{Z}] \otimes \mathbb{C}[y_n, n > 0]$:

$$f_n \mapsto x_n,\ \ h_n \mapsto -2\sum_{m\in\mathbb{Z}} x_{m+n}\partial x_m + \delta_{n<0} y_{-n} + \delta_{n>0} 2nK\partial y_n + \delta_{n,0} J,$$

$$e_n \mapsto -\sum_{m,k\in\mathbb{Z}} x_{k+m+n}\partial x_k \partial x_m + \sum_{k>0} y_k \partial x_{-k-n} + \\ + 2K\sum_{m>0} m\partial y_m \partial x_{m-n} + (Kn + J)\partial x_{-n}.$$

This module is irreducible if and only if $K \neq 0$. If we let $K = 0$ and quotient out the submodule generated by $y_m, m > 0$ then the factormodule is irreducible if and only if $J \neq 0$ (cf. [F1]). This construction has been generalized for all affine Lie algebras in [C2] providing a realization of imaginary Verma modules.

7. Second free field realization

There is another way to correct the formulas obtained in Example 6.1, which leads to the construction of Wakimoto modules [W].

Denote $a_n = \partial x_n$, $a_n^* = x_{-n}$ and consider formal power series

$$a(z) = \sum_{n\in\mathbb{Z}} a_n z^{-n-1}, \quad a^*(z) = \sum_{n\in\mathbb{Z}} a_n^* z^{-n}.$$

Series $a(z)$ and $a^*(z)$ are called *formal distributions.* It is easy to see that $[a_n, a_m^*] = \delta_{n+m,0}$ and all other products are zero. The formulas in Example 6.1 can be rewritten as follows:

$$e(z) \mapsto a(z),\ \ h(z) \mapsto -2a^*(z)a(z),\ \ f(z) \mapsto -a^*(z)^2 a(z),$$

where $g(z) = \sum_{n\in\mathbb{Z}} g_n z^{-n-1}$ for $g \in \{e, f, h\}$. This realization is not well-defined since the annihilation and creation operators are in a wrong order. It becomes well-defined after the application of two anti-involutions described above. Then the formulas read:

$$f(z) \mapsto a(z), \;\; h(z) \mapsto 2a(z)a^*(z), \;\; f(z) \mapsto -a(z)a^*(z)^2,$$

where a_n and a^*_n have the following meaning now $a_n = x_n$, $a^*_n = -\partial x_{-n}$. This is our quotient of the imaginary Verma module.

A different approach was suggested by Wakimoto ([W]) who introduced the *normal ordering.* Denote

$$a(z)_- = \sum_{n<0} a_n z^{-n-1}, \;\; a(z)+ = \sum_{n\geq 0} a_n z^{-n-1}$$

and define the normal ordering as follows

$$: a(z)b(z) := a(z)_- b(z) + b(z)a_+(z).$$

Let now

$$a_n = \{ \begin{array}{cc} x_n, & n<0 \\ \partial x_n, & n\geq 0, \end{array} \quad a^*_n = \{ \begin{array}{cc} x_{-n}, & n\leq 0 \\ -\partial x_{-n}, & n>0, \end{array} \quad b_m = \{ \begin{array}{cc} m\partial y_m, & m\geq 0 \\ y_{-m}, & m<0. \end{array}$$

Here $[a_n, a^*_m] = [b_n, b_m] = \delta_{n+m,0}$.

Theorem 7.1 ([W]). *The formulas*

$$c \mapsto K, \;\; e(z) \mapsto a(z), \;\; h(z) \mapsto -2 : a^*(z)a(z) : +b(z),$$

$$f(z) \mapsto - : a^*(z)^2 a(z) : +K\partial_z a^*(z) + a^*(z)b(z)$$

define the second free field realization *of the affine $sl(2)$ acting on the space* $\mathbb{C}[x_n, n \in \mathbb{Z}] \otimes \mathbb{C}[y_m, m > 0]$.

These modules are celebrated *Wakimoto modules.* They were defined for an arbitrary affine Lie algebra by Feigin and Frenkel [FF1], [FF2]. Wakimoto modules have many interesting features and still far from a complete understanding. Generically they are isomorphic to Verma modules. Wakimoto modules can be viewed as infinite twistings of Verma modules. They are related to the *semi-infinite cohomology* introduced in [Fe] and can be obtained by a *semi-infinite induction* [FF2], [V1], [V2].

8. Intermediate Wakimoto modules

So far we considered two extreme cases of Borel subalgebras in the affine Lie algebras: standard and natural. But if the rank of $\mathfrak{G}$ is more than 1 then the corresponding affine algebra has other conjugacy classes of Borel subalgebras. Geometrically they should produce some "semi-semi-infinite" flag varieties. On the other hand it should be possible to associate to each such Borel subalgebra a boson type realization by rearranging the annihilation and creation operators as it was done in the first and the second free field realizations. For affine Lie algebras of type $A_n^{(1)}$ associated with $sl(n+1)$ this has been accomplished in [CF], where a series of boson type realizations was constructed depending on the parameter $0 \leq r \leq n$. If $r = n$ this construction coincides with the construction of Wakimoto modules. On the other hand when $r = 0$ the obtained representation gives a Fock space realization described in [C2].

Let $0 \leq r \leq n$, $\gamma \in \mathbb{C}^*$, $k = \gamma^2 - (r+1)$. Let H_i, E_i, F_i, $i = 1, \dots, n$ be the standard basis for $\mathfrak{G} = sl(n+1)$. Denote $X_m = t^m \otimes X$ for $X, Y \in \mathfrak{G}$ and $m \in \mathbb{Z}$. Let $\{\alpha_1, \dots, \alpha_n\}$ be a basis for Δ^+, the positive set of roots for $\mathfrak{G}$, such that $H_i = \check{\alpha}_i$ and let Δ_r be the root system with basis $\{\alpha_1, \dots, \alpha_r\}$ ($\Delta_r = \emptyset$, if $r = 0$) of the Lie subalgebra $\mathfrak{G}_r = sl(r+1)$. Denote by $\mathcal{H}_r$ a Cartan subalgebra of $\mathfrak{G}_r$ spanned by H_i, $i = 1, \dots, r$. Set $\mathcal{H}_0 = 0$, $\tilde{\mathcal{H}_r} = \mathcal{H}_r \oplus \mathbb{C}c \oplus \mathbb{C}d$.

Denote by E_{im}, F_{im}, H_{im}, $i = 1, \dots, n$, $m \in \mathbb{Z}$, the generators of the loop algebra corresponding to $\mathfrak{G}$.

Let $\hat{\mathfrak{a}}$ be the infinite dimensional Heisenberg algebra with generators $a_{ij,m}$, $a^*_{ij,m}$, and $\mathbf{1}$, $1 \leq i \leq j \leq n$ and $m \in \mathbb{Z}$, subject to the relations

$$\begin{aligned} [a_{ij,m}, a_{kl,n}] &= [a^*_{ij,m}, a^*_{kl,n}] = 0, \\ [a_{ij,m}, a^*_{kl,n}] &= \delta_{ik}\delta_{jl}\delta_{m+n,0}\mathbf{1}, \\ [a_{ij,m}, \mathbf{1}] &= [a^*_{ij,m}, \mathbf{1}] = 0. \end{aligned}$$

This algebra acts on $\mathbb{C}[x_{ij,m} | i, j, m \in \mathbb{Z},\ 1 \leq i \leq j \leq n]$ by

$$a_{ij,m} \mapsto \begin{cases} \partial/\partial x_{ij,m} & \text{if} \quad m \geq 0, \text{ and } j \leq r \\ x_{ij,m} & \text{otherwise,} \end{cases}$$

$$a^*_{ij,m} \mapsto \begin{cases} x_{ij,-m} & \text{if} \quad m \leq 0, \text{ and } j \leq r \\ -\partial/\partial x_{ij,-m} & \text{otherwise.} \end{cases}$$

and $\mathbf{1}$ acts as an identity. Hence we have an $\hat{\mathfrak{a}}$-module generated by v such that

$$a_{ij,m}v = 0, \quad m \geq 0 \text{ and } j \leq r, \quad a^*_{ij,m}v = 0, \quad m > 0 \text{ or } j > r.$$

Let $\hat{\mathfrak{a}}_r$ denote the subalgebra generated by $a_{ij,m}$ and $a^*_{ij,m}$ and $\mathbf{1}$, where $1 \le i \le j \le r$ and $m \in \mathbb{Z}$. If $r = 0$, we set $\hat{\mathfrak{a}}_r = 0$.

Let $((\alpha_i|\alpha_j))$ be the Cartan matrix for $sl(n+1)$ and let

$$\mathfrak{B}_{ij} := (\alpha_i|\alpha_j)(\gamma^2 - \delta_{i>r}\delta_{j>r}(r+1) + \frac{r}{2}\delta_{i,r+1}\delta_{j,r+1}).$$

Let $\hat{\mathfrak{b}}$ be the Heisenberg Lie algebra with generators b_{im}, $1 \le i \le n$, $m \in \mathbb{Z}$, $\mathbf{1}$, and relations $[b_{im}, b_{jp}] = m\,\mathfrak{B}_{ij}\delta_{m+p,0}\mathbf{1}$ and $[b_{im}, \mathbf{1}] = 0$.

For each $1 \le i \le n$ fix $\lambda_i \in \mathbb{C}$ and let $\lambda = (\lambda_1, \dots, \lambda_n)$. Then the algebra $\hat{\mathfrak{b}}$ acts on the space $\mathbb{C}[y_{i,m}|i, m \in \mathbb{N}^*, 1 \le i \le n]$ by

$$b_{i0} \mapsto \lambda_i, \quad b_{i,-m} \mapsto \mathbf{e}_i \cdot \mathbf{y}_m, \quad b_{im} \mapsto m\mathbf{e}_i \cdot \frac{\partial}{\partial \mathbf{y}_m} \quad \text{for} \quad m > 0$$

and $\mathbf{1} \mapsto 1$. Here

$$\mathbf{y}_m = (y_{1m}, \cdots, y_{nm}), \quad \frac{\partial}{\partial \mathbf{y}_m} = \left(\frac{\partial}{\partial y_{1m}}, \cdots, \frac{\partial}{\partial y_{nm}}\right)$$

and $\mathbf{e}_i$ are vectors in $\mathbb{C}^n$ such that $\mathbf{e}_i \cdot \mathbf{e}_j = \mathfrak{B}_{ij}$ where $\cdot$ means the usual dot product.

For any $1 \le i \le j \le n$, we define

$$a^*_{ij}(z) = \sum_{n \in \mathbb{Z}} a^*_{ij,n} z^{-n}, \quad a_{ij}(z) = \sum_{n \in \mathbb{Z}} a_{ij,n} z^{-n-1}$$

and

$$b_i(z) = \sum_{n \in \mathbb{Z}} b_{in} z^{-n-1}.$$

Then

$$[b_i(z), b_j(w)] = \mathfrak{B}_{ij}\partial_w \delta(z-w), \quad [a_{ij}(z), a^*_{kl}(w)] = \delta_{ik}\delta_{jl}\mathbf{1}\delta(z-w),$$

where

$$\delta(z-w) = z^{-1}\sum_{n \in \mathbb{Z}} \left(\frac{z}{w}\right)^n.$$

is the *formal delta function.*

Set

$$a_{ij}(z)_+ = a_{ij}(z), \quad a_{ij}(z)_- = 0$$
$$a^*_{ij}(z)_+ = 0, \quad a^*_{ij}(z)_- = a^*_{ij}(z),$$

if $j > r$.

Denote $\mathbb{C}[\mathbf{x}] = C[x_{ij,m}|i, j, m \in \mathbb{Z}, 1 \le i \le j \le n]$ and $C[\mathbf{y}] = C[y_{i,m}|i, m \in \mathbb{N}^*, 1 \le i \le n]$.

Remark 8.1. *Note that*

$$: a_{ij}(z)a^*_{kl}(z) := \sum_{m\in\mathbb{Z}}\left(\sum_{n\in\mathbb{Z}} : a_{ij,n}a^*_{kl,m-n} :\right) z^{-m-1}$$

is well defined on $\mathbb{C}[\mathbf{x}]\otimes\mathbb{C}[\mathbf{y}])[[z,z^{-1}]]$ *for all* $l>r$ *or if* $l\le r$ *and* $j\le r$.

Let $\mathfrak{b}_r$ be the Borel subalgebra of $\tilde{\mathfrak{G}}$ corresponding to a parabolic subalgebra of $\mathfrak{G}$ whose semisimple part of the Levi factor is $\mathfrak{G}_r$.

Fix $\tilde\lambda \in \tilde{\mathcal{H}}^*$ and let $M_r(\tilde\lambda)$ be the Verma type module associated with $\mathfrak{b}_r$ and $\tilde\lambda$. When $r=n$ this module is a standard Verma module while in the case $r=0$ we get an imaginary Verma module. Denote by $v_{\tilde\lambda}$ the generator of $M_r(\tilde\lambda)$.

Let $\tilde\lambda_r = \tilde\lambda|_{\tilde{\mathcal{H}}_r}$. The module $M_r(\tilde\lambda)$ contains a $\tilde{\mathfrak{G}}_r$-submodule $M(\tilde\lambda_r) = U(\tilde{\mathfrak{G}}_r)(v_{\tilde\lambda})$ which is isomorphic to the standard Verma module for $\tilde{\mathfrak{G}}_r$. If $\tilde\lambda(c)\neq 0$ then the submodule structure of $M_r(\tilde\lambda)$ is completely determined by the submodule structure of $M(\tilde\lambda_r)$ [C1], [FS].

Define

$$E_i(z) = \sum_{n\in\mathbb{Z}} E_{in}z^{-n-1},\quad F_i(z) = \sum_{n\in\mathbb{Z}} F_{in}z^{-n-1},\quad H_i(z) = \sum_{n\in\mathbb{Z}} H_{in}z^{-n-1},$$

$$1\le i\le n.$$

Theorem 8.2 ([CF]). *Let* $\lambda\in\mathfrak{H}^*$ *and set* $\lambda_i=\lambda(H_i)$. *The generating functions*

$$F_i(z)\mapsto a_{ii} + \textstyle\sum_{j=i+1}^n a_{ij}a^*_{i+1,j},$$

$$\begin{aligned} H_i(z)\mapsto 2 : a_{ii}a^*_{ii} : &+ \textstyle\sum_{j=1}^{i-1}\left(: a_{ji}a^*_{ji} : - : a_{j,i-1}a^*_{j,i-1} :\right)\\ &+ \textstyle\sum_{j=i+1}^{n}\left(: a_{ij}a^*_{ij} : - : a_{i+1,j}a^*_{i+1,j} :\right) + b_i, \end{aligned}$$

$$\begin{aligned} E_i(z)\mapsto : a^*_{ii}&\left(\textstyle\sum_{k=1}^{i-1} a_{k,i-1}a^*_{k,i-1} - \sum_{k=1}^{i} a_{ki}a^*_{ki}\right) : + \textstyle\sum_{k=i+1}^{n} a_{i+1,k}a^*_{ik}\\ &- \textstyle\sum_{k=1}^{i-1} a_{k,i-1}a^*_{ki} - a^*_{ii}b_i - \left(\delta_{i>r}(r+1) + \delta_{i\le r}(i+1) - \gamma^2\right)\partial a^*_{ii}, \end{aligned}$$

$$c\mapsto \gamma^2 - (r+1)$$

define a representation on the Fock space $\mathbb{C}[\mathbf{x}]\otimes\mathbb{C}[\mathbf{y}]$. *In the above* a_{ij}, a^*_{ij} *and* b_i *denotes* $a_{ij}(z)$, $a^*_{ij}(z)$ *and* $b_i(z)$ *respectively.*

This boson type realization of $\tilde{sl}(n+1)$ depends on the parameter $0 \leq r \leq n$ and defines a module structure on the Fock space $\mathbb{C}[\mathbf{x}] \otimes \mathbb{C}[\mathbf{y}]$ which is called an *intermediate Wakimoto module.* Denote it by $W_{n,r}(\lambda, \gamma)$ and consider a $\tilde{\mathfrak{G}}_r$-submodule $W = U(\tilde{\mathfrak{G}}_r)(v_{\tilde{\lambda}}) \simeq W_{r,r}(\lambda, \gamma)$ of $W_{n,r}(\lambda, \gamma)$. Then W is isomorphic to the Wakimoto module $W_{\lambda(r),\tilde{\gamma}}$ ([FF2]), where $\lambda(r) = \lambda|_{\mathcal{H}_r}$, $\tilde{\gamma} = \gamma^2 - (r+1)$. Since generically Wakimoto modules are isomorphic to Verma modules, intermediate Wakimoto modules provide a realization for generic Verma type modules.

We see that the same geometrical object $G((t))/\mathfrak{B}_{nat}$ induced three different constructions that depend on normal ordering and the interpretation of the bosons: first free field realization (imaginary Verma modules), second free field realization (Wakimoto modules) and intermediate Wakimoto modules. Same variety will produce twisted Wakimoto modules and twisted intermediate Wakimoto modules, but imaginary Verma modules do not admit twisting. Note that Wakimoto modules are obtained from the Verma modules by infinite twisting (cf. [FF2]) and hence can not be obtained from the regular flag variety. Similarly, intermediate Wakimoto modules are infinite twistings of the corresponding Verma type modules and can not be obtained from geometrical objects with other Borel subgroups. These flag "varieties" will produce Verma type modules and contragradient Verma type modules.

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CONTACT INFORMATION

V. Futorny IME-USP, Caixa Postal 66281, Sao Paulo, CEP 05315-970, Brazil
E-Mail: futorny@ime.usp.br

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