

Gorenstein matrices

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Dedicated to Yu.A. Drozd on the occasion of his 60th birthday

ABSTRACT. Let $A = (a_{ij})$ be an integral matrix. We say that A is $(0, 1, 2)$ -matrix if $a_{ij} \in \{0, 1, 2\}$. There exists the Gorenstein $(0, 1, 2)$ -matrix for any permutation σ on the set $\{1, \dots, n\}$ without fixed elements. For every positive integer n there exists the Gorenstein cyclic $(0, 1, 2)$ -matrix A_n such that $inx A_n = 2$.

If a Latin square \mathcal{L}_n with a first row and first column $(0, 1, \dots, n-1)$ is an exponent matrix, then $n = 2^m$ and \mathcal{L}_n is the Cayley table of a direct product of m copies of the cyclic group of order 2. Conversely, the Cayley table \mathcal{E}_m of the elementary abelian group $G_m = (2) \times \dots \times (2)$ of order 2^m is a Latin square and a Gorenstein symmetric matrix with first row $(0, 1, \dots, 2^m - 1)$ and

$$\sigma(\mathcal{E}_m) = \begin{pmatrix} 1 & 2 & 3 & \dots & 2^m - 1 & 2^m \\ 2^m & 2^m - 1 & 2^m - 2 & \dots & 2 & 1 \end{pmatrix}.$$

1. Introduction

Gorenstein rings appeared in a paper by D. Gorenstein published in 1952 [9]. In [1] H. Bass wrote: “After writing this paper I discovered from Professor Serre that these rings have been encountered by Grothendick the latter having christened in his setting by the fact that a certain module of differentials is locally free of rank one”. (See, also [2]).

Let \mathcal{O} be a Dedekind ring with a field of fractions K , and let Λ be an \mathcal{O} -order in a finite dimensional separable K -algebra A (see [6]). In this

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case it is natural to consider Λ -lattices, i.e., finitely generated \mathcal{O} -torsion free Λ -modules.

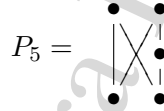
Noncommutative Gorenstein \mathcal{O} -orders appeared first in [8], (see Definition and Proposition 6.1). An \mathcal{O} -order Λ is left Gorenstein if and only if the injective dimension of Λ as a left Λ -module is 1 ($\mathcal{O} \neq K$). Definition and Proposition 6.1 of [8] shows that Λ is left Gorenstein if and only if it is right Gorenstein.

Given a Λ -lattice M , a sublattice N of M is called **pure** if M/N is \mathcal{O} -torsion free.

The following theorem is proved in [10]:

An \mathcal{O} -order Λ is Gorenstein if and only if each left Λ -lattice is isomorphic to a pure sublattice of a free Λ -lattice.

In the [20] K. Nishida gives an example of a $(0, 1)$ -order $\Lambda(P_5)$ associated with the finite poset



such that $\text{inj dim } \Lambda(P_5) = 2$ and $\text{gl. dim } \Lambda(P_5) = \infty$.

Let Λ be a Gorenstein order. If Λ has the additional property that every \mathcal{O} -order containing Λ is also Gorenstein, then Λ is called a Bass order. The following inclusions are easily verified:

$$\begin{aligned} (\text{maximal orders}) &\subseteq (\text{hereditary orders}) \subseteq \\ &\subseteq (\text{Bass orders}) \subseteq (\text{Gorenstein orders}) \end{aligned}$$

(see [6, §37]).

Denote by $\mu_\Lambda(X)$ the minimal number of generators of a finitely generated Λ -module X . The following theorem is proved in [22] (see also [6, Theorem 37.17]).

Let Λ be an \mathcal{O} -order such that $\mu_\Lambda(I) \leq 2$ for each left ideal I of Λ . Then Λ is a Bass order.

Obviously, the Z -order

$$\begin{pmatrix} \mathbb{Z} & 4\mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} \end{pmatrix}$$

is a Bass order, because for every left ideal J we have $\mu_\Lambda(I) \leq 2$, (see also [3]).

In [11] H. Fujita studies an interesting class of algebras which is closely related to tiled orders over discrete valuation rings.

Tiled orders over a discrete valuation rings appeared first in [23] (see also [13, 14]). The Gorenstein condition for exponent matrices of tiled orders is formulated in [15]. Note that the notion of an exponent matrix appeared, first, in the English translation of [24].

A finite directed graph without multiple arrows and multiple loops is called **simply laced**.

Denote by $M_n(B)$ the ring of all $n \times n$ matrices over a ring B .

An integer matrix $\mathcal{E} = (\alpha_{ij}) \in M_n(\mathbb{Z})$ is called

- an exponent matrix if $\alpha_{ij} + \alpha_{jk} \geq \alpha_{ik}$ and $\alpha_{ii} = 0$ for all i, j, k ;
- a reduced exponent matrix if $\alpha_{ij} + \alpha_{ji} > 0$ for all $i, j: i \neq j$.

Recall that a ring A is called a **tiled order** if it is a prime Noetherian semiperfect semidistributive ring with nonzero Jacobson radical (see [4, 5]).

Theorem 1.1. *Each tiled order A is isomorphic to a prime ring of the following form:*

$$A = \begin{pmatrix} \mathcal{O} & \pi^{\alpha_{12}}\mathcal{O} & \dots & \pi^{\alpha_{1n}}\mathcal{O} \\ \pi^{\alpha_{21}}\mathcal{O} & \mathcal{O} & \dots & \pi^{\alpha_{2n}}\mathcal{O} \\ \dots & \dots & \dots & \dots \\ \pi^{\alpha_{n1}}\mathcal{O} & \pi^{\alpha_{n2}}\mathcal{O} & \dots & \mathcal{O} \end{pmatrix},$$

where $n \geq 1$, \mathcal{O} is a discrete valuation ring with a prime element π , and the α_{ij} are integers with $\alpha_{ij} + \alpha_{jk} \geq \alpha_{ik}$ for all i, j, k ($\alpha_{ii} = 0$ for all i).

We shall use the following notation: $A = \{\mathcal{O}, \mathcal{E}(A)\}$, where $\mathcal{E}(A) = (\alpha_{ij})$ is the exponent matrix of A , i.e., $A = \sum_{i,j=1}^n e_{ij}\pi^{\alpha_{ij}}\mathcal{O}$, where the e_{ij} are the matrix units. If a tiled order is reduced, then $\alpha_{ij} + \alpha_{ji} > 0$ for $i, j = 1, \dots, n, i \neq j$, i.e., $\mathcal{E}(A)$ is reduced.

Note that with every reduced tiled order A we associate the following notions (see [4, 5]):

- 1) the reduced exponent matrix $\mathcal{E}(A)$;
- 2) the quiver $Q(A)$ which coincides with the quiver $Q(\mathcal{E}(A))$;
- 3) the width $w(A)$ which coincides with the width $w(\mathcal{E}(A))$ of $\mathcal{E}(A)$;
- 4) the index of A ($\text{inx } A$).

By definition $\text{inx } \mathcal{E}(A) = \text{inx } A$.

Let $\mathcal{E} = (\alpha_{ij})$ be a reduced exponent matrix. Set $\mathcal{E}^{(1)} = (\beta_{ij})$, where $\beta_{ij} = \alpha_{ij}$ for $i \neq j$ and $\beta_{ii} = 1$ for $i = 1, \dots, n$, and $\mathcal{E}^{(2)} = (\gamma_{ij})$, where $\gamma_{ij} = \min_{1 \leq k \leq n} (\beta_{ik} + \beta_{kj})$.

Theorem 1.2. [17]. *The matrix $[Q] = \mathcal{E}^{(2)} - \mathcal{E}^{(1)}$ is the adjacency matrix of the strongly connected simply laced quiver $Q = Q(\mathcal{E})$.*

A strongly connected simply laced quiver is called admissible if it is the quiver of a reduced exponent matrix.

Theorem 1.3. [18]. *An arbitrary strongly connected simply laced quiver Q with a loop in every vertex is admissible.*

The main concept of this paper is the notion of a **Gorenstein matrix**.

A reduced exponent matrix $\mathcal{E} = (\alpha_{ij}) \in M_n(\mathbb{Z})$ shall be called **Gorenstein** if there exists a permutation σ of $\{1, 2, \dots, n\}$ such that $\alpha_{ik} + \alpha_{k\sigma(i)} = \alpha_{i\sigma(i)}$ for $i, k = 1, \dots, n$.

The permutation σ is denoted by $\sigma(\mathcal{E})$. Notice that $\sigma(\mathcal{E})$ of a Gorenstein matrix \mathcal{E} has no cycles of length 1.

A Gorenstein matrix \mathcal{E} is called **cyclic** if $\sigma(\mathcal{E})$ is a cycle.

A simply laced quiver Q shall be called **Gorenstein** if $Q = Q(\mathcal{E})$ for a Gorenstein matrix \mathcal{E} .

2. Examples

Let σ be a permutation of $\{1, 2, \dots, n\}$. Then $P_\sigma = \sum_{i=1}^n e_{i\tau(i)}$ is called a **permutation matrix** (here e_{ij} stand for the matrix units).

In [18, Theorem 4.5] the following theorem was proved:

Theorem 2.1. *The adjacency matrix of the quiver of a cyclic Gorenstein matrix \mathcal{E} with permutation $\sigma = \sigma(\mathcal{E})$ is a sum of some powers of the permutation matrix P_σ .*

We will give examples of Gorenstein matrices.

Examples.

I. The $(n \times n)$ -matrix

$$H_n = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & 0 & \dots & 0 & 0 \\ 1 & 1 & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & 1 & 1 & \dots & 0 & 0 \\ 1 & 1 & 1 & \dots & 1 & 0 \end{pmatrix}$$

is a Gorenstein cyclic matrix with permutation

$$\sigma = \sigma(H_n) = \begin{pmatrix} 1 & 2 & \dots & n-1 \\ n & 1 & \dots & n \end{pmatrix}$$

For the adjacency matrix $[Q(H_n)]$ we have that $[Q(H_n)] = P_{\sigma^{n-1}}$.

Remark. The matrices H_n appeared in the theorem by Michler [19], which we state below after giving some notation.

Let \mathcal{O} be a (possibly non-commutative) discrete valuation ring with the division ring of fractions \mathcal{D} and let \mathcal{M} be its unique maximal ideal. Denote by $H_n(\mathcal{O})$ the subring of the matrix ring $M_n(\mathcal{D})$ of the form

$$H_n(\mathcal{O}) = \begin{pmatrix} \mathcal{O} & \mathcal{O} & \dots & \mathcal{O} \\ \mathcal{M} & \mathcal{O} & \dots & \mathcal{O} \\ \dots & \dots & \dots & \dots \\ \mathcal{M} & \mathcal{M} & \dots & \mathcal{O} \end{pmatrix}.$$

Clearly, the ring $H_n(\mathcal{O})$ is hereditary and $\mathcal{E}(H_n(\mathcal{O})) = H_n$.

Theorem 2.2. [19]. *Every semiprime semiperfect Noetherian hereditary ring is Morita equivalent to the finite direct product of division rings and some rings of the form $H_m(\mathcal{O})$.*

II.

The $(2m \times 2m)$ -matrix

$$G_{2m} = \begin{vmatrix} H_m & H_m^{(1)} \\ H_m^{(1)} & H_m \end{vmatrix}$$

is Gorenstein with permutation

$$\sigma(G_{2m}) = \begin{pmatrix} 1 & 2 & \dots & m & m+1 & m+2 & \dots & 2m \\ m+1 & m+2 & \dots & 2m & 1 & 2 & \dots & m \end{pmatrix}.$$

If $m = 1$ then

$$[Q(G_2)] = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = E + P_\tau,$$

where

$$\tau = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}.$$

In general case, $[Q(G_{2m})] = P_{\tau^{m-1}} + P_{\tau^{2m-1}}$, where

$$\tau = \begin{pmatrix} 1 & 2 & \dots & 2m \\ 2m & 1 & \dots & 2m-1 \end{pmatrix}$$

is a cycle and $\text{inx } G_{2m} = 2$.

III.

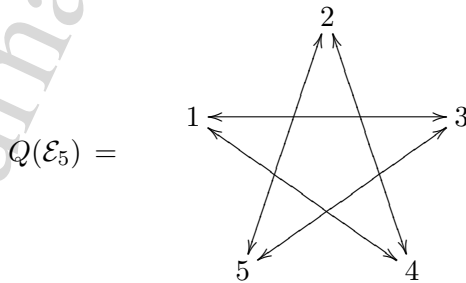
The matrix

$$\mathcal{E}_5 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 2 & 1 & 1 & 2 & 0 \end{pmatrix}$$

is cyclic Gorenstein with permutation

$$\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 1 & 2 & 3 & 4 \end{pmatrix}$$

and $[Q(\mathcal{E}_5)] = P_{\tau^2} + P_{\tau^3}$.



IV.

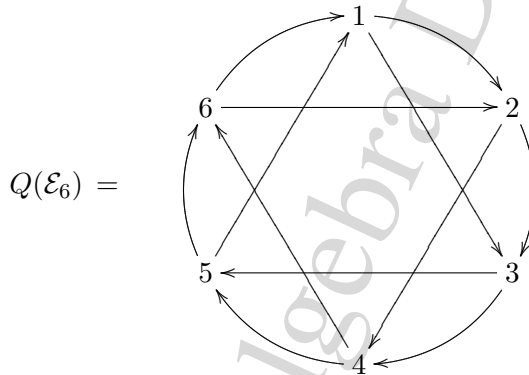
The matrix

$$\mathcal{E}_6 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 2 & 1 & 2 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 2 & 1 & 2 & 1 & 2 & 0 \end{pmatrix}$$

is cyclic Gorenstein with permutation

$$\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 1 & 2 & 3 & 4 & 5 \end{pmatrix}$$

and $[Q(\mathcal{E}_6)] = P_{\tau^4} + P_{\tau^5}$.

**V.**

The matrix

$$\Gamma_6 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 4 & 0 & 4 & 4 & 3 & 3 \\ 4 & 0 & 0 & 4 & 2 & 2 \\ 4 & 0 & 0 & 0 & 1 & 1 \\ 3 & 0 & 1 & 2 & 0 & 3 \\ 3 & 0 & 1 & 2 & 3 & 0 \end{pmatrix}$$

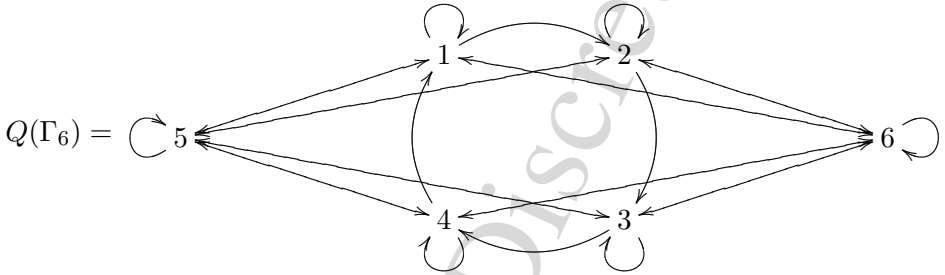
is Gorenstein with permutation

$$\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 4 & 1 & 6 & 5 \end{pmatrix}.$$

Note that

$$[Q(\Gamma_6)] = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 1 \end{pmatrix}$$

is not a multiple doubly stochastic matrix. We have that



Definition 2.3. [5]. Two exponent matrices $\mathcal{E} = (\alpha_{ij})$ and $\Theta = (\theta_{ij})$ shall be called **equivalent** if they can be obtained from each other by transformations of the following two types :

- (1) subtracting an integer from the i -th row with simultaneous adding it to the i -th column;
- (2) simultaneous interchanging of two rows and the equally numbered columns.

Proposition 2.4. [5]. Suppose that $\mathcal{E} = (\alpha_{ij})$ and $\Theta = (\theta_{ij})$ are exponent matrices and Θ is obtained from \mathcal{E} by a transformation of type (1). Then $[Q(\mathcal{E})] = [Q(\Theta)]$. If \mathcal{E} is a reduced Gorenstein exponent matrix with permutation $\sigma(\mathcal{E})$, then Θ is also reduced Gorenstein with $\sigma(\Theta) = \sigma(\mathcal{E})$.

Proposition 2.5. [5]. Under transformations of the second type the adjacency matrix $[\tilde{Q}]$ of $Q(\Theta)$ changes according to the formula: $[\tilde{Q}] = P_\tau^T [Q] P_\tau$, where $[Q] = [Q(\mathcal{E})]$. If \mathcal{E} is Gorenstein then Θ is also Gorenstein and for the new permutation π we have: $\pi = \tau^{-1} \sigma \tau$, i.e., $\sigma(\Theta) = \tau^{-1} \sigma(\mathcal{E}) \tau$.

Theorem 2.6. Any Gorenstein $(0,1)$ -matrix is equivalent either to H_n or to G_{2m} .

The proof follows from [16, Theorem 2.1].

Corollary 2.7. Any cyclic Gorenstein $(0,1)$ -matrix is equivalent to a matrix H_n and $\text{inx } H_n = 1$. Conversely, if $\text{inx } \mathcal{E} = 1$, where \mathcal{E} is a reduced exponent matrix, then \mathcal{E} is equivalent to H_n .

Let σ be a permutation of $\{1, \dots, n\}$ without fixed elements. There exists a Gorenstein matrix \mathcal{E}_σ such that $\sigma(\mathcal{E}_\sigma) = \sigma$ (see [5], Theorem 6.3). The Gorenstein quiver $Q(\mathcal{E}_\sigma)$ shall be called the quiver associated with the permutation σ .

Definition 2.8. *A permutation σ without fixed elements shall be called exceptional if the Gorenstein quiver associated with σ is unique, up to isomorphism.*

Proposition 2.9. *The permutation $\sigma = (12)(345)$ is exceptional.*

Proof. We describe all Gorenstein matrices $\mathcal{E}_\sigma = (\alpha_{ij})$. We can assume that $\alpha_{11} = \alpha_{12} = \alpha_{13} = \alpha_{14} = \alpha_{15} = 0$. So, $\alpha_{12} = \alpha_{22} = \alpha_{32} = \alpha_{42} = \alpha_{52} = 0$.

We have the following system of linear equations for elements of \mathcal{E}_σ :

$$\begin{cases} \alpha_{12} = \alpha_{23} + \alpha_{31} = \alpha_{24} + \alpha_{41} = \alpha_{25} + \alpha_{51} \\ \alpha_{31} = \alpha_{24} = \alpha_{34} = \alpha_{35} + \alpha_{54} \\ \alpha_{41} = \alpha_{25} = \alpha_{43} + \alpha_{35} = \alpha_{45} \\ \alpha_{51} = \alpha_{23} = \alpha_{53} = \alpha_{54} + \alpha_{43} \end{cases}$$

It is easy to see that

$$\mathcal{E}_\sigma = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 4\alpha & 0 & 2\alpha & 2\alpha & 2\alpha \\ 2\alpha & 0 & 0 & 2\alpha & \alpha \\ 2\alpha & 0 & \alpha & 0 & 2\alpha \\ 2\alpha & 0 & 2\alpha & \alpha & 0 \end{pmatrix}.$$

and

$$\mathcal{E}^{(1)} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 4\alpha & 1 & 2\alpha & 2\alpha & 2\alpha \\ 2\alpha & 0 & 1 & 2\alpha & \alpha \\ 2\alpha & 0 & \alpha & 1 & 2\alpha \\ 2\alpha & 0 & 2\alpha & \alpha & 1 \end{pmatrix},$$

$$\mathcal{E}^{(2)} = \begin{pmatrix} 2 & 0 & 1 & 1 & 1 \\ 4\alpha & 2 & 2\alpha + 1 & 2\alpha + 1 & 2\alpha + 1 \\ 2\alpha + 1 & 1 & 2 & 2\alpha & \alpha + 1 \\ 2\alpha + 1 & 1 & \alpha + 1 & 2 & 2\alpha \\ 2\alpha + 1 & 1 & 2\alpha & \alpha + 1 & 2 \end{pmatrix}$$

therefore,

$$[Q(\mathcal{E})] = \begin{pmatrix} 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 \end{pmatrix}.$$

□

3. Gorenstein $(0, 1, 2)$ -matrices

Denote the ring of all square $n \times n$ -matrices over the integers \mathbb{Z} by $M_n(\mathbb{Z})$. Let $A \in M_n(\mathbb{Z})$.

Definition 3.1. A matrix $A = (a_{ij})$ shall be called a $(0, 1, 2)$ -matrix if $a_{ij} \in \{0, 1, 2\}$.

Theorem 3.2. For any permutation σ on $\{1, \dots, n\}$ without fixed elements there exists a Gorenstein $(0, 1, 2)$ -matrix.

Proof. Let $\sigma : i \rightarrow \sigma(i)$ be a permutation on $\{1, \dots, n\}$ without fixed elements and $\mathcal{E}_\sigma = (\alpha_{ij})$ be the following $(0, 1, 2)$ -matrix:

- $\alpha_{ii} = 0$ and $\alpha_{i\sigma(i)} = 2$ for $i = 1, \dots, n$;
- $\alpha_{ij} = 1$ for $i \neq j$ and $i \neq \sigma(i)$ ($i, j = 1, \dots, n$).

Obviously, \mathcal{E}_σ is a Gorenstein matrix with permutation σ . □

Let σ be an arbitrary permutation on $\{1, \dots, n\}$ without fixed elements and \mathcal{E}_σ be a Gorenstein $(0, 1, 2)$ -matrix as in Theorem 3.2. Denote $P_\sigma = \sum_{i=1}^n e_{i\sigma(i)}$ the permutation matrix of σ . It is easy to see that $[Q(\mathcal{E}_\sigma)] = U_n - P_\sigma$.

We will show how one can represent the matrix $[Q(\mathcal{E}_\sigma)]$ as a sum of permutation matrices.

Let $\sigma_1, \dots, \sigma_{n-1}$ be the permutations: $\sigma_k(i) = \sigma(i) + k \pmod{n}$. Obviously, $\sigma_k(i) \neq \sigma_m(i)$ for $k \neq m$ and $[Q(\mathcal{E}_\sigma)] = \sum_{k=1}^{n-1} P_{\sigma_k}$.

Examples.

I.

Let

$$\mathcal{E}_3 = \begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 2 & 2 & 0 \end{pmatrix}$$

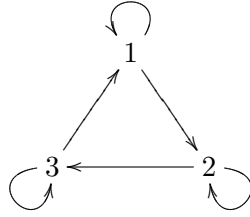
be the Gorenstein matrix with permutation

$$\sigma = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}.$$

Obviously,

$$\mathcal{E}_3^{(2)} = \begin{pmatrix} 1 & 1 & 0 \\ 2 & 1 & 1 \\ 3 & 2 & 1 \end{pmatrix} \text{ and } [Q(\mathcal{E}_3)] = E + P_{\sigma^2}.$$

Thus, $Q(\mathcal{E}_3)$ has the following form:



II.

Let

$$\mathcal{E}_4 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 2 & 1 & 2 & 0 \end{pmatrix}$$

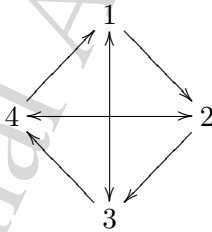
be the Gorenstein matrix with permutation

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix}.$$

Obviously,

$$\mathcal{E}_4^{(2)} = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 2 & 1 & 2 & 1 \\ 2 & 1 & 1 & 1 \\ 3 & 2 & 2 & 1 \end{pmatrix} \text{ and } [Q(\mathcal{E}_4)] = P_{\sigma^2} + P_{\sigma^3} = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix}.$$

Hence, $Q(\mathcal{E}_4)$ has the following form:



III.

Let

$$\mathcal{E}_5 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 2 & 1 & 1 & 2 & 0 \end{pmatrix}$$

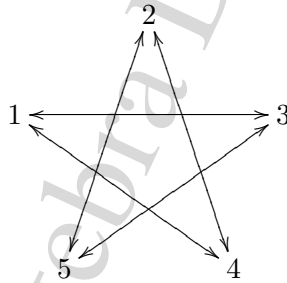
be the Gorenstein matrix with permutation

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 1 & 2 & 3 & 4 \end{pmatrix}.$$

Obviously

$$\mathcal{E}_5^{(2)} = \begin{pmatrix} 1 & 0 & 1 & 1 & 0 \\ 2 & 1 & 1 & 2 & 1 \\ 2 & 1 & 1 & 1 & 1 \\ 2 & 1 & 1 & 1 & 0 \\ 2 & 2 & 2 & 2 & 1 \end{pmatrix} \text{ and } [Q(\mathcal{E}_5)] = \begin{pmatrix} 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{pmatrix}.$$

So, $Q(\mathcal{E}_5)$ has the following form



and $[Q(\mathcal{E}_5)] = P_{\sigma^2} + P_{\sigma^3}$.

IV.

Let

$$\mathcal{E}_6 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 2 & 1 & 1 & 1 & 2 & 0 \end{pmatrix}$$

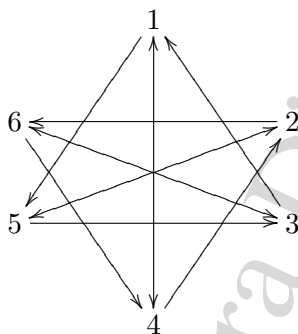
be the Gorenstein matrix with permutation

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 1 & 2 & 3 & 4 & 5 \end{pmatrix}.$$

Obviously,

$$\mathcal{E}_6^{(2)} = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 2 & 1 & 1 & 1 & 2 & 1 \\ 2 & 1 & 1 & 1 & 1 & 1 \\ 2 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 \\ 2 & 1 & 2 & 2 & 2 & 1 \end{pmatrix} \quad [Q(\mathcal{E}_6)] = P_{\sigma^2} + P_{\sigma^3} = \begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{pmatrix}.$$

We have that $Q(\mathcal{E}_6)$ is of the following form:



In the general case we have that

$$\mathcal{E}_n = \begin{pmatrix} 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 2 & 0 & 1 & 1 & \dots & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & \dots & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & \dots & 1 & 1 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & 0 & 0 & 0 & \dots & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & \dots & 1 & 0 & 0 \\ 2 & 1 & 1 & 1 & \dots & 1 & 2 & 0 \end{pmatrix}$$

is a Gorenstein matrix with permutation

$$\sigma(\mathcal{E}_n) = \sigma = \begin{pmatrix} 1 & 2 & \dots & n \\ n & 1 & \dots & n-1 \end{pmatrix}.$$

It is easy to show that $[Q(\mathcal{E}_n)] = P_{\sigma^2} + P_{\sigma^3}$.

Theorem 3.3. *For every positive integer n there exists a Gorenstein cyclic $(0, 1, 2)$ -matrix \mathcal{E}_n such that $\text{inx } \mathcal{E}_n = 2$.*

4. Latin squares and Cayley tables of elementary abelian 2-groups

A Latin square of order n is a square matrix with rows and columns each of which is a permutation of $S = \{s_1, \dots, s_n\}$.

Every Latin square is a Cayley table of a finite quasigroup. In particular, the Cayley table of a finite group is a Latin square. We take $S = \{0, 1, \dots, n - 1\}$.

Examples.

I.

The Latin square

$$\mathcal{L}_4 = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 3 & 2 \\ 3 & 2 & 0 & 1 \\ 2 & 3 & 1 & 0 \end{pmatrix}$$

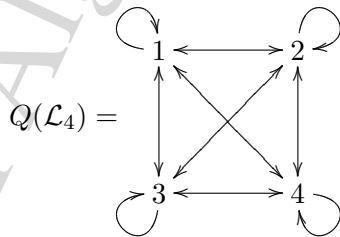
is a Gorenstein matrix with permutation

$$\sigma = \sigma(\mathcal{L}_4) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 1 & 2 \end{pmatrix}$$

and

$$[Q(\mathcal{L}_4)] = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix}.$$

Obviously, $[Q(\mathcal{L}_4)] = E + P_{\sigma^2} + P_{\sigma^3}$ and



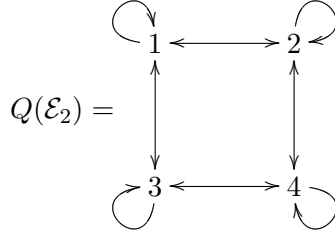
II.

The Latin square

$$\mathcal{E}_2 = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 3 & 2 \\ 2 & 3 & 0 & 1 \\ 3 & 2 & 1 & 0 \end{pmatrix}$$

is the Cayley table of the Klein four-group and is a Gorenstein matrix with permutation $\sigma(\mathcal{E}) = (14)(23)$. By Propositions 2.4 and 2.5 the matrices \mathcal{E}_2 and \mathcal{L}_4 are non-equivalent.

$$\mathcal{E}_2^{(1)} = \begin{pmatrix} 1 & 1 & 2 & 3 \\ 1 & 1 & 3 & 2 \\ 2 & 3 & 1 & 1 \\ 3 & 2 & 1 & 1 \end{pmatrix}; \quad \mathcal{E}_2^{(2)} = \begin{pmatrix} 2 & 2 & 3 & 3 \\ 2 & 2 & 3 & 3 \\ 3 & 3 & 2 & 2 \\ 3 & 3 & 2 & 2 \end{pmatrix}.$$



We introduce the following notation:

$$\mathcal{E}_0 = (0), \quad \mathcal{E}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \mathcal{E}_2 = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 3 & 2 \\ 2 & 3 & 0 & 1 \\ 3 & 2 & 1 & 0 \end{pmatrix},$$

$$U_n \in M_n(\mathbb{Z}) \text{ and } U_n = \begin{pmatrix} 1 & 1 & \dots & 1 \\ 1 & 1 & \dots & 1 \\ \dots & \dots & \dots & \dots \\ 1 & 1 & \dots & 1 \end{pmatrix}, \quad X_{k-1} = 2^{k-1}U_{2^{k-1}};$$

$$\mathcal{E}_k = \begin{pmatrix} \mathcal{E}_{k-1} & \mathcal{E}_{k-1} + X_{k-1} \\ \mathcal{E}_{k-1} + X_{k-1} & \mathcal{E}_{k-1} \end{pmatrix} \text{ for } k = 1, 2, \dots.$$

Obviously, \mathcal{E}_k is a Gorenstein matrix with permutation

$$\sigma = \sigma(\mathcal{E}_k) = \begin{pmatrix} 1 & 2 & \dots & k \\ 2^k & 2^k - 1 & \dots & 1 \end{pmatrix}.$$

Main Theorem. *Suppose that a Latin square \mathcal{L}_n with a first row and a first column $(01 \dots n-1)$ is an exponent matrix. Then $n = 2^m$ and $\mathcal{L}_n = \mathcal{E}_m$ is the Cayley table of a direct product of m copies of the cyclic group of order 2.*

Conversely, the Cayley table \mathcal{E}_m of the elementary abelian group $G_m = (2) \times \dots \times (2)$ (m factors) of order 2^m is the Latin square and the Gorenstein symmetric matrix with the first row $(0, 1, \dots, 2^m - 1)$ and

$$\sigma(\mathcal{E}_m) = \begin{pmatrix} 1 & 2 & 3 & \dots & 2^m - 1 & 2^m \\ 2^m & 2^m - 1 & 2^m - 2 & \dots & 2 & 1 \end{pmatrix}.$$

The second part of this theorem was proved in [21, Section 4].

Lemma 4.1. *Let $\mathcal{L}_n = (\alpha_{ij})$ be defined as above. Then*

$$|i - j| \leq \alpha_{ij} \leq i + j - 2.$$

Proof. Obviously, $\alpha_{1i} + \alpha_{ij} \geq \alpha_{1j}$ and $\alpha_{ij} \geq j - 1 - (i - 1) = j - i$. Analogously, $\alpha_{ij} + \alpha_{j1} \geq \alpha_{i1}$ and $\alpha_{ij} \geq i - 1 - (j - 1) = i - j$, i.e., $\alpha_{ij} \geq |i - j|$. We have $\alpha_{i1} + \alpha_{1j} \geq \alpha_{ij}$ and $\alpha_{ij} \leq i + j - 2$. \square

Lemma 4.2. *The last row of \mathcal{L}_n is $(n - 1, n - 2, \dots, 1)$.*

Proof. We have that $\alpha_{n1} = n - 1$ by the definition of \mathcal{L}_n . By Lemma 4.1 we have $\alpha_{ni} \geq n - i$. So, $\alpha_{n2} = n - 2$, $\alpha_{n3} = n - 3$ and $\alpha_{nn} = 0$. \square

Corollary 4.3. *The last column of \mathcal{L}_n is $(n - 1, n - 2, \dots, 1)^T$, where T is the transpose.*

Lemma 4.4. *Let $\mathcal{L}_n = (\alpha_{ij})$ be defined as above. Then*

$$|i + j - (n + 1)| \leq |n - 1 - \alpha_{ij}|.$$

Proof. By Lemma 4.2 and Corollary 4.3, we have $\alpha_{ij} \leq \alpha_{in} + \alpha_{nj} = (n - i) + (n - j) = 2n - (i + j)$. By Lemma 4.1 $\alpha_{ij} \leq i + j - 2$. From first inequality we have

$$\alpha_{ij} - (n - 1) \leq n + 1 - (i + j).$$

From second inequality we have

$$\alpha_{ij} - (n - 1) \leq (i + j) - (n + 1).$$

So

$$|(i + j) - (n + 1)| \leq |\alpha_{ij} - (n - 1)|.$$

\square

Corollary 4.5. *An integer n is even.*

Proof. By Lemma 4.4 an integer $n - 1$ appears on the (i, j) -th position if $|(n - 1) - (n - 1)| \geq |(i + j) - (n + 1)|$, i.e., $i + j = n + 1$. Hence, the second diagonal has the following form: $(n - 1, \dots, n - 1)$. If n is odd then for $i = j = \frac{n+1}{2}$ we have $\alpha_{ii} = n - 1$. This is a contradiction. \square

Now we will prove the Main Theorem.

- for every k the number of the blocks \mathcal{E}_k ($\mathcal{E}_k \neq \mathcal{E}$) is always even.

$$\mathcal{E}_k = \begin{pmatrix} \mathcal{E}_{k-1} & \mathcal{E}_{k-1} + X_{k-1} \\ \mathcal{E}_{k-1} + X_{k-1} & \mathcal{E}_{k-1} \end{pmatrix}.$$

The base of induction was proved.

Assume \mathcal{E} contains n_k blocks \mathcal{E}_k (each of the order 2^k) on the main block diagonal.

$$\mathcal{E}_k = \begin{pmatrix} \mathcal{E}_{k-1} & \mathcal{E}_{k-1} + 2^{k-1}U_k \\ \mathcal{E}_{k-1} + 2^{k-1}U_k & \mathcal{E}_{k-1} \end{pmatrix}.$$

From $\alpha_{ij} = 2^k$ follows that $|i - j| \leq 2^k$. It is easy to see $\alpha_{ij} = 2^k$ for $j = 2^k + i$ or $i = 2^k + j$. More precisely

$$\alpha_{t \cdot 2^{k+1} + i, t \cdot 2^{k+1} + 2^k + i} = 2^k$$

$$\alpha_{t \cdot 2^{k+1} + 2^k + i, t \cdot 2^{k+1} + i} = 2^k$$

for $t > 0, i = 1, 2, \dots, 2^k$.

It is easy to see that $\alpha_{j, 2^k + j} = 2^k$ for $j = 1, 2, \dots, 2^k$. Analogously, $\alpha_{2^k + i, i} = 2^k$ for $i = 1, 2, \dots, 2^k$ and

$$\alpha_{t \cdot 2^{k+1} + i, t \cdot 2^{k+1} + 2^k + i} = 2^k,$$

$$\alpha_{t \cdot 2^{k+1} + 2^k + i, t \cdot 2^{k+1} + i} = 2^k$$

for $t > 0, i = 1, 2, \dots, 2^k$.

Besides, the blocks \mathcal{E}_k unify pairwise into blocks Y_i , such that in each row and in each column of whose there is an element 2^k . If the number n_k of blocks is odd, then it is impossible to allocate the numbers in the last 2^k rows and columns of the matrix \mathcal{E} in such a way that they appear in each row and each column only once (in the block \mathcal{E}_{k-1} they cannot be allocated). Therefore, the number n_k is even.

From the inequalities

$$\begin{aligned} 2^k &\leq \alpha_{t \cdot 2^{k+1} + i, t \cdot 2^{k+1} + 2^k + j} \leq \alpha_{t \cdot 2^{k+1} + i, t \cdot 2^{k+1} + j} + \alpha_{t \cdot 2^{k+1} + j, t \cdot 2^{k+1} + 2^k + j} = \\ &= \alpha_{t \cdot 2^{k+1} + i, t \cdot 2^{k+1} + j} + 2^k < 2^{k+1}; \end{aligned}$$

$$\begin{aligned} 2^k &\leq \alpha_{t \cdot 2^{k+1} + 2^k + i, t \cdot 2^{k+1} + j} \leq \alpha_{t \cdot 2^{k+1} + 2^k + i, t \cdot 2^{k+1} + i} + \alpha_{t \cdot 2^{k+1} + i, t \cdot 2^{k+1} + j} = \\ &= 2^k + \alpha_{t \cdot 2^{k+1} + i, t \cdot 2^{k+1} + j} < 2^{k+1} \end{aligned}$$

for $j = 1, 2, \dots, 2^k$ we have

$$Y_i = \begin{pmatrix} \mathcal{E}_k & \mathcal{E}_{12} \\ \mathcal{E}_{21} & \mathcal{E}_k \end{pmatrix},$$

where \mathcal{E}_{12} , \mathcal{E}_{21} are Latin squares.

Since $2^k U_k \leq \mathcal{E}_{12} < 2^{k+1}$ then $0 \leq \mathcal{E}_{12} - 2^k U_k < 2^k$, and $\mathcal{E}_{12} - 2^k U_k$ is a Latin square on the set $\{1, 2, \dots, 2^k - 1\}$.

Since $2^k U_k \leq \mathcal{E}_{21} < 2^{k+1}$ then $0 \leq \mathcal{E}_{21} - 2^k U_k < 2^k$, and $\mathcal{E}_{21} - 2^k U_k$ is a Latin square on the set $\{1, 2, \dots, 2^k - 1\}$.

Furthermore

$$\begin{aligned} \alpha_{t, 2^{k+1}+i, t, 2^{k+1}+2^{k+1}} &\leq \\ &\leq \alpha_{t, 2^{k+1}+i, t, 2^{k+1}+1} + \alpha_{t, 2^{k+1}+1, t, 2^{k+1}+2^{k+1}} = \\ &= (i-1) + 2^k = 2^k + i - 1 \end{aligned}$$

$$\begin{aligned} \alpha_{t, 2^{k+1}+2^k+i, t, 2^{k+1}+1} &\leq \\ &\leq \alpha_{t, 2^{k+1}+2^k+i, t, 2^{k+1}+i} + \alpha_{t, 2^{k+1}+i, t, 2^{k+1}+1} = \\ &= 2^k + (i-1) = 2^k + i - 1 \end{aligned}$$

for all $i = 1, 2, \dots, 2^k$. For $i = 1, 2, \dots, 2^k$ we have that

$$\alpha_{t, 2^{k+1}+i, t, 2^{k+1}+2^{k+1}} = 2^k + i - 1$$

$$\alpha_{t, 2^{k+1}+2^k+i, t, 2^{k+1}+1} = 2^k + i - 1$$

for all $i = 1, 2, \dots, 2^k$. So the first row (the first column) of the matrices \mathcal{E}_{12} and \mathcal{E}_{21} has the following form $(0, 1, \dots, 2^k - 1) \ ((0, 1, \dots, 2^k - 1)^T)$.

By induction we have $\mathcal{E}_{12} - 2^k U_k = \mathcal{E}_{21} - 2^k U_k = \mathcal{E}_k$ and blocks

$$\mathcal{E}_{k+1} = \begin{pmatrix} \mathcal{E}_k & \mathcal{E}_k + X_k \\ \mathcal{E}_k + X_k & \mathcal{E}_k \end{pmatrix}$$

are on the main diagonal.

Thus n_k is divided on two. The following blocks

$$\mathcal{E}_{k+1} = \begin{pmatrix} \mathcal{E}_k & \mathcal{E}_k + 2^k U \\ \mathcal{E}_k + 2^k U & \mathcal{E}_k \end{pmatrix}$$

are on the diagonal of the matrix \mathcal{E} . The induction hypothesis is proved.

We have shown that the number n_k of the blocks \mathcal{E}_k on the diagonal \mathcal{E} is $n_{k+1} = \frac{n_k}{2}$ and $n = 2^k n_k$. For some $k = m$ we have $n_m = 1$. Then $n = 2^m$ and $\mathcal{E} = \mathcal{E}_m$.

$$\mathcal{E} = \mathcal{E}_m = \begin{pmatrix} \mathcal{E}_{m-1} & \mathcal{E}_{m-1} + X_{m-1} \\ \mathcal{E}_{m-1} + X_{m-1} & \mathcal{E}_{m-1} \end{pmatrix}.$$

□

Remark. *Example 1* shows, that for the Main Theorem the condition for a Latin square to be with the first row and the first column of the form $(01 \dots n - 1)$ is essential.

5. Admissible quivers

Let P be an arbitrary poset. A subset of P is called a **chain** if any two of its elements are related. A subset of P is called an **antichain** if no two distinct elements of the subset are related.

We shall denote a chain of n elements by CH_n and an antichain of n elements by ACH_n .

Theorem 5.1. [7], [12] *Given a poset, the minimal number of disjoint chains which together contain all elements of P is equal to the maximal number of elements in an antichain, if this number is finite.*

Definition 5.2. [12] *The maximal number $w(P)$ of elements in an antichain of P is called the width of P .*

With a reduced exponent $(0, 1)$ -matrix \mathcal{E} we associate the partially ordered set

$$P_{\mathcal{E}} = \{1, \dots, n\}$$

with the relation \leq defined by the formula: $i \leq j \Leftrightarrow \alpha_{ij} = 0$.

Conversely, with any finite poset $P = \{1, \dots, n\}$ we relate the reduced $(0, 1)$ -matrix $\mathcal{E}_P = (\lambda_{ij})$ by the following way: $\lambda_{ij} = 0 \Leftrightarrow i \leq j$, otherwise, $\lambda_{ij} = 1$.

Let CH_m be a linearly ordered set CH_n . Then $Q(\mathcal{E}_{CH_n})$ is a simple cycle with n vertices. Let

$$ACH_n = \left\{ \begin{array}{cccccc} 1 & 2 & \dots & n-1 & n \\ \bullet & \bullet & \dots & \bullet & \bullet \end{array} \right\}$$

be an antichain of width n . Then $Q(\mathcal{E}_{ACH_n})$ is a complete simply laced quiver with n vertices.

Theorem 5.3. *For every natural m ($1 \leq m \leq n$, $m \neq n-1$) there exists an admissible quiver with n vertices and exactly m loops.*

Let $P = ACH_m \cup CH_{n-m}$ ($m \neq n-1$). Obviously, $Q(\mathcal{E}_P)$ is an admissible quiver with n vertices and exactly m loops.

Remark. It is easy to see that there is no admissible quiver with $n-1$ loops.

Theorem 5.4. *There exists an exponent matrix M_k such that $\text{inx } M_k = k$ for any $1 \leq k \leq n$.*

Proof. We saw that $\text{inx } H_n = 1$ and $\text{inx } \mathcal{E}_{ACH_n} = n$. Let τ be the cyclic permutation:

$$\tau = \begin{pmatrix} 1 & 2 & \dots & n \\ n & 1 & \dots & n-1 \end{pmatrix}.$$

Then the quiver Q_k ($k \geq 2$) with the adjacency matrix: $E + P_\tau + \dots + P_{\tau^{k-1}}$ is admissible by Theorem 1.3. So, there exists an exponent matrix M_k and $\text{inx } M_k = k$. \square

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