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Direct decompositions of artinian modules related to formations of groups

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Abstract. We survey direct decompositions of artinian modules over group rings into two summands where all the chief factors of the first are $\mathfrak{X}$–central and all the chief factors of the other is $\mathfrak{X}$–eccentric, where $\mathfrak{X}$ is a certain formation of finite groups.

1. Introduction

Artinian modules are one of oldest objects of study in Algebra. The study of artinian modules over group rings began in the second half of the past century and was mainly stimulated by questions of the theory of (soluble) groups. Although the underlying group of the group ring considered can have a very complicated structure, the study of the action of the group on the module play a major role and provides in a natural way the existence of some direct decomposition of modules, which gives a good information for the group itself. Probably, a celebrated result due to H. Fitting, known as Fitting's lemma, is one of the first result on which one of these direct decompositions appear. We formulate it in the following form.

Theorem (Fitting) *Let A be an RG–module of finite composition length, where R is a ring and G is a finite nilpotent group. Then $A = C \oplus E$, where the RG–chief factors U/V of E (respectively of E) satisfy $G = C_G(U/V)$ (respectively, $G \neq C_C(U/V)$). In particular, $A = C_A(G) \oplus A(\omega RG)$ provided $\Pi(G) \cap \Pi(A) = \emptyset$.*

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This raises the subordinated questions both of finding out complements of the upper RG–hypercenter and that of studying certain extensions of modules, which are near to modules of finite composition length. These problem are found very important applications in the study both of groups and modules with finiteness conditions. Moreover they are also connected with the question of the existence of complements for some residuals in groups.

Now we formulate the necessary concepts in their more general form.

A class $\mathfrak{X}$ of groups is said to be *a formation* if and only if it satisfies the following conditions:

(F1) If $G \in \mathfrak{X}$ and H is a normal subgroup of G, then $G/H \in \mathfrak{X}$.

(F2) If H_1 and H_2 are normal subgroups of G such that G/H_1 and $G/H_2 \in \mathfrak{X}$, then $G/H_1 \cap H_2\mathfrak{X}$.

Let R be a ring, G a group, $\mathfrak{X}$ a class of groups and A an RG–module. As suggested above, if $B \leq C$ are RG–submodules of A, the factor C/B is said to be $\mathfrak{X}$*–central* (respectively, $\mathfrak{X}$*–eccentric*) if $G/C_G(C/B) \in \mathfrak{X}$ (respectively, $G/C_G(C/B) \notin \mathfrak{X}$). To rule out these factors, we similarly define

$$\mathfrak{X}C_{RG}(A) = \{a \in A \mid G/C_G(aRG) \in \mathfrak{X}\}.$$

As $\mathfrak{X}$ is a formation of groups, then $\mathfrak{X}C_{RG}(A)$ is an RG–submodule of A called the $\mathfrak{X}$*–center of* A (more precisely, *the* $\mathfrak{X} - RG$*–center of* A). Proceeding in a similar way to that we did for groups, we construct the upper $\mathfrak{X}$–central series of the module A as

$$\{0\} = A_0 \leq A_1 \leq \ \cdots \ A_\alpha \leq A_{\alpha+1} \leq \ \cdots \leq A_\gamma$$

where $A_1 = \mathfrak{X}C_{RG}(A)$, $A_{\alpha+1}/A_\alpha = \mathfrak{X}C_{RG}(A/A_\alpha)$, for all ordinals $\alpha < \gamma$ and $\mathfrak{X}C_{RG}(A/A_\gamma) = \{0\}$. The last term $A_\gamma = HZ_{\mathfrak{X}-RG}(A)$ of this series is called *the upper* $\mathfrak{X}$*–hypercenter of* A (or *the* $\mathfrak{X} - RG$*–hypercenter*) and the other terms A_α, *the* $\mathfrak{X}$*–hypercenters of* A. If $A = A_\gamma$, then A is said to be $\mathfrak{X}$*–hypercentral* ($\mathfrak{X}$*–nilpotent*, if γ is finite). If $\mathfrak{X} = \mathfrak{I}$ and $\mathfrak{X} = \mathfrak{F}$, we have the RG–center $\zeta_{RG}(A)$ of A, the upper RG–hypercenter $\zeta_{RG}^{\infty}(A)$ of A, the FC–center $FC_{RG}(A)$ of A and the upper FC–hypercenter $FC_{RG}^{\infty}(A)$ of A.

On the other hand, an RG–submodule C of A is said to be $\mathfrak{X} - RG$*–hypereccentric* if it has an ascending series

$$\{0\} \ = C_0 \leq C_1 \leq \ \cdots \ C_\gamma \leq C_{\alpha+1} \leq \ \cdots \ C_\gamma = C$$

of RG–submodules of A such that each factor $C_{\alpha+1}/C_\alpha$ is an $\mathfrak{X}$–eccentric simple RG–module, for every $\alpha < \gamma$.

Given a formation $\mathfrak{X}$ of groups, we say that *the RG–module A has the $\mathfrak{X}$–decomposition* or, more precisely, *the $\mathfrak{X} - RG$–decomposition*) if the following equality holds

$$A = HZ_{\mathfrak{X}-RG}(A) \bigoplus HE_{\mathfrak{X}-RG}(A).$$

It is worth to mentioning that $HE_{\mathfrak{X}-RG}(A)$ is the unique maximal $\mathfrak{X} - RG$–hypereccentric RG–submodule of A. For, let B an $\mathfrak{X}-RG-$hypereccentric RG–submodule of A, $E = HE_{\mathfrak{X}-RG}(A)$. If $(B+E)/E$ is non-zero, it includes a non-zero simple RG–submodule U/E. Since $(B + E)/E \cong B/(B \cap E)$, U/E is RG–isomorphic to some simple RG–factor of B and it follows that $G/C_G(U/E) \notin \mathfrak{X}$. On the other hand, $(B + E)/E \leq A/E \cong HZ_{\mathfrak{X}-RG}(A)$, that is $G/C_G(U/E) \in \mathfrak{X}$. This contradiction shows that $B \leq E$. Hence $HE_{\mathfrak{X}-RG}(A)$ includes every $\mathfrak{X} - RG-$hypereccentric RG–submodule and, in particular, it is unique, as claimed.

If $\mathfrak{X} = \mathfrak{I}$, the decomposition is simply called *the $\mathfrak{Z}$–decomposition* whereas if $\mathfrak{X} = \mathfrak{F}$, we called it *the $\mathfrak{F}$–decomposition*.

Since we are discussing the existence of the $\mathfrak{X}$–decomposition for modules over $\mathfrak{X}C$–hypercentral groups, we will need to define concepts in groups that are similar to the above ones.

If G is a group and $x \in G$, we put $x^G = \{g^{-1}xg \mid g \in G\}$; clearly, $C_G(x^G)$ is normal in G. Let now $\mathfrak{X}$ be a class of groups, and define *the $\mathfrak{X}$–center* of G as

$$\mathfrak{X}C(G) = \{x \in G \mid G/C_G(x^G) \in \mathfrak{X}\}.$$

If $\mathfrak{X}$ is a formation of groups, then $\mathfrak{X}C(G)$ is a characteristic subgroup of G and G is said to be *an $\mathfrak{X}C$–group* if the equality $G = \mathcal{X}C(G)$ holds. If $\mathfrak{X} = \mathfrak{I}$ is the class of all identity groups, then $\mathfrak{X}C(G) = \zeta(G)$ is the ordinary center of G whereas, if $\mathfrak{X} = \mathfrak{F}$ is the class of all finite groups, then $\mathfrak{X}C(G) = FC(G)$ is precisely the FC–center of G. From this subgroup, we may construct *the upper $\mathfrak{X}$–central series* of G as

$$\langle 1 \rangle = C_0 \leq C_1 \leq \cdots \leq C_\alpha \leq C_{\alpha+1} \leq \cdots C_\gamma,$$

where $C_1 = \mathfrak{X}C(G)$, $C_{\alpha+1}/C_\alpha = \mathfrak{X}C(G/C_\alpha)$, for an $\alpha < \gamma$, and $\mathfrak{X}C(G/C_\gamma) = \langle 1 \rangle$. The last term C_γ of this series is called *the upper $\mathfrak{X}$–hypercenter of G* and denoted by $HZ_{\mathfrak{X}}(G)$. If $G = C_\gamma$, then G is said to be *$\mathfrak{X}$–hypercentral* and, if γ is finite, *$\mathfrak{X}$–nilpotent*. Once again, if $\mathfrak{X} = \mathfrak{I}$ or $\mathfrak{X} = \mathfrak{F}$, $HZ_{\mathfrak{X}}(G) = \zeta^\infty(G)$ or $= FC^\infty(G)$ are the upper hypercenter or upper FC–hypercenter of G.

2. Some results

The first result on the existence of the $\mathfrak{Z}$–decomposition for infinite modules was obtained by Hartley and Tomkinson [3].

Theorem 1. *Let G be a locally nilpotent group and let A be a $\mathbb{Z}G$–module. If A is $\mathbb{Z}$–periodic and the p–component of A has finite special rank for every prime p, then A has the $\mathfrak{Z}$–decomposition.*

In this case, $G/C_G(A)$ is actually hypercentral and every p–component of A is an artinian $\mathbb{Z}$–module. Artinian modules are the most classical extension of modules, having finite composition series. Thus, we naturally come to the question of the existence of the $\mathfrak{Z}$–decomposition for artinian modules. The question was solved by Zaitsev [9], which showed the following result.

Theorem 2. *Let G be a hypercentral group and let A be a $\mathbb{Z}G$–module. If A is an artinian $\mathbb{Z}G$–module, then A has the $\mathfrak{Z}$–decomposition.*

Note that this result can be easily extended to artinian DG–module, where D is a Dedekind domain and G is a hypercentral group.

The following natural step was the consideration of the question of the existence of the $\mathfrak{F}$–decomposition for artinian modules. We notice first that in this study the underlying group considered has to be FC–hypercentral instead of hypercentral. A first result in this direction was obtained by Zaitsev [10, 11] as follows.

Theorem 3. *Let G be a hyperfinite locally soluble group and A a $\mathbb{Z}G$–module. If A is an artinian $\mathbb{Z}G$–module, then A has the $\mathfrak{F}$–decomposition.*

Theorem 4. *Let G be an FC–hypercentral group and let A be a $\mathbb{Z}G$–module. If A has finite composition $\mathbb{Z}G$–series, then A has the $\mathfrak{F}$–decomposition.*

The next result is due to Duan [1], who showed a partial solution in considering a special type of FC–hypercentral groups.

Theorem 5. *Let G be a locally soluble group having an ascending series of normal subgroups, every factor of which is finite or cyclic and let A be a $\mathbb{Z}G$–module. If A is an artinian $\mathbb{Z}G$–module, then A has the $\mathfrak{F}$–decomposition.*

The final solution for the formation $\mathfrak{F}$ was obtained by Kurdachenko, Petrenko and Subbotin [5].

Theorem 6. *Let G be a locally soluble FC–hypercentral group, D a Dedekind domain and A a DG–module. If A is an artinian DG–module, then A has the $\mathfrak{F}$–decomposition.*

In Module Theory, modules having a finite composition series are the algebraic objects more analogous to finite groups. For these modules, we can obtain the following result.

Theorem 7. *Let $\mathfrak{X}$ be a formation of finite groups, G an $\mathfrak{X}$–hypercentral group, D a Dedekind domain and A a DG–module. If A has a finite composition DG–series, then A has the $\mathfrak{X}$–decomposition.*

Since G is FC–hypercentral, A has the $\mathfrak{F}$–decomposition, that is $A = B \oplus C$, where $B = HZ_{\mathfrak{F}-DG}(A)$ and $C = HE_{\mathfrak{F}-DG}(A)$. Since B has a finite composition series every factor of which is $\mathfrak{F}$–central, $G/C_G(B)$ is finite. By [7], B has the $\mathfrak{X}$–decomposition and then A has the $\mathfrak{X}$–decomposition too.

Note that in this result, $\mathfrak{X}$ is an arbitrary formation of finite groups.

The question of the existence of the $\mathfrak{F}$–decomposition has been an important partial case of the general case and its solution has allowed to obtain the solution for many important formations $\mathfrak{X}$. At this point, it is needed to split the general study into two cases according to $\mathfrak{F} \leq \mathfrak{X}$ or $\mathfrak{X}$ is a proper formation of finite groups.

A formation $\mathfrak{X}$ is said to be *overfinite* if it satisfies the following conditions:

(i) if $G \in \mathfrak{X}$ and H is a normal subgroup of G of finite index, then $H \in \mathfrak{X}$.

(ii) if G is a group, H is a normal subgroup of finite index of G and $H \in \mathfrak{X}$, then $G \in \mathfrak{X}$.

(iii) $\mathfrak{I} \leq \mathfrak{X}$.

Clearly, an overfinite formation always contains $\mathfrak{F}$. The most important examples of these formations are polycyclic groups, Chernikov groups, soluble minimax groups, soluble groups of finite special rank and soluble groups of finite section rank For locally soluble FC–hypercentral groups, the existence of the $\mathfrak{X}$–decomposition for an overfinite formation $\mathfrak{X}$ in an artinian DG–module A was also showed by Kurdachenko, Petrenko and Subbotin in [6], who proved the next result.

Theorem 8. *Let D be a Dedekind domain, G a locally soluble FC–hypercentral group and A an artinian DG–module. If $\mathfrak{X}$ is an overfinite formation of groups, then A has the $\mathfrak{X}$–decomposition.*

Corollary 1. *If G is a locable soluble FC–hypercentral group and D is a Dedekind domain, then an artinian DG–module A has the $\mathfrak{X}$-decomposition for the following formations $\mathfrak{X}$:*

(1) $\mathfrak{F}$, formation of all finite groups;
(2) $\mathfrak{P}$, formation of all polycyclic groups;
(3) $\mathfrak{C}$, formation of all Chernikov groups;
(4) $\mathfrak{S}_2$, formation of all soluble minimax groups;
(5) $\mathfrak{S}^{\wedge}$, formation of all soluble groups of finite special rank; and
(6) $\mathfrak{S}_0$, formation of all soluble groups of finite section rank.

Since every formation $\mathfrak{X}$ includes $\mathfrak{F}$, every FC–hypercentral group is likewise $\mathfrak{X}$–hypercental. Thus, the next natural step is to consider that the underlyin group is $\mathfrak{X}$–hypercentral and study artinian DG–modules.

A formation $\mathfrak{X}$ of finite groups is said to be *infinitely hereditary concerning a class of groups* $\mathfrak{Y}$ if it satisfies the following condition:

(IH) whenever an $\mathfrak{Y}$–group G belongs to the class $\mathbf{R}\mathfrak{X}$, then every finite factor–group of G belongs to $\mathfrak{X}$.

Many important formations of finite groups are infinitely hereditary concerning the class of FC–hypercentral groups, as for example:

(1) $\mathfrak{A}\cap\mathfrak{F}$, the formation of finite abelian groups,

(2) $\mathfrak{N}_c\cap\mathfrak{F}$, the formation of finite nilpotent groups of class at most c,

(3) $\mathfrak{S}_d\cap\mathfrak{F}$, the formation of finite soluble groups of derived length at most d,

(4) $\mathfrak{S}\cap\mathfrak{F}$, the formation of finite soluble groups,

(5) $\mathfrak{B}(n)\cap\mathfrak{F}$, the formation of finite groups of exponent dividing n among others. Moreover all these five examples and

(6) $\mathfrak{N}\cap\mathfrak{F}$, the formation of finite nilpotent groups,

(7) $\mathfrak{U}\cap\mathfrak{F}$, finite supersoluble groups

are infinitely hereditary concerning both the classes of the FC–groups and hyperfinite groups.

Main results for these examples were obtained by Kurdachenko, Otal and Subbotin [4]

Theorem 9. *Let D be a Dedekind domain, $\mathfrak{X}$ a formation of finite groups, G an infinite $\mathfrak{X}$–hypercentral group, A an artinian DG–module. If $\mathfrak{X}$ is infinitely hereditary concerning the class of FC–hypercentral groups, then A has the $\mathfrak{X}$–decomposition.*

Corollary 2. *Let D be a Dedekind domain, $\mathfrak{X}$ a formation of finite groups, G an infinite $\mathfrak{X}$–hypercentral group, A an artinian DG–module. Then A has the $\mathfrak{X}$–decomposition for the formations $\mathfrak{X}=\mathfrak{A}\cap\mathfrak{F}$, $\mathfrak{N}_c\cap\mathfrak{F}$, $\mathfrak{S}_d\cap\mathfrak{F}$, $\mathfrak{S}\cap\mathfrak{F}$ and $\mathfrak{B}(n)\cap\mathfrak{F}$.*

Corollary 3. *Let D be a Dedekind domain, $\mathfrak{X}$ a formation of finite groups, G an infinite $\mathfrak{X}$–hypercentral group, A an artinian DG–module.*

If G is an FC–group, then A has the $\mathfrak{X}$–decomposition for the formations $\mathfrak{X} = \mathfrak{A} \cap \mathfrak{F}$, $\mathfrak{N}_c \cap \mathfrak{F}$, $\mathfrak{S}_d \cap \mathfrak{F}$, $\mathfrak{S} \cap \mathfrak{F}$, $\mathfrak{B}(n) \cap \mathfrak{F}$, $\mathfrak{N} \cap \mathfrak{F}$ and $\mathfrak{U} \cap \mathfrak{F}$.

Corollary 4. *Let D be a Dedekind domain, $\mathfrak{X}$ a formation of finite groups, G an infinite $\mathfrak{X}$–hypercentral group, A an artinian DG–module. If G is a hyperfinite group, then A has the $\mathfrak{X}$–decomposition for the formations $\mathfrak{X} = \mathfrak{A} \cap \mathfrak{F}$, $\mathfrak{N}_c \cap \mathfrak{F}$, $\mathfrak{S}_d \cap \mathfrak{F}$, $\mathfrak{S} \cap \mathfrak{F}$, $\mathfrak{B}(n) \cap \mathfrak{F}$, $\mathfrak{N} \cap \mathfrak{F}$ and $\mathfrak{U} \cap \mathfrak{F}$.*

Corollary 5. *Let D be a Dedekind domain, $\mathfrak{X}$ a formation of finite groups, G an infinite $\mathfrak{X}$–hypercentral group, A an artinian DG–module. If G is a Chernikov group, then A has the $\mathfrak{X}$–decomposition.*

Corollary 6. *Let D be a Dedekind domain, $\mathfrak{X}$ a formation of finite groups, G an infinite $\mathfrak{X}$–hypercentral group, A an artinian DG–module. If G is finitely generated, then A has the $\mathfrak{X}$–decomposition.*

The other classical generalization of modules with finite composition series are the noetherian modules. Here the situation is quite different as we cannot consider direct decomposition any longer for a noetherian module, as the following example shows. Let $A = \langle u \rangle \times \langle v \rangle$ be a free abelian group of rank 2. We construct the split extension G of A by a finite cyclic group $\langle g \rangle$ of order 3, where the action is given by: $u^g = v$ and $v^g = u^{-1}v^{-1}$. Then every non-identity G–invariant subgroup of A has finite index. In particular, the $\mathbb{Z}\langle g \rangle$–module A is noetherian. However A is directly indecomposable and has central and non-central G–chief factors.

Despite of this, Robinson [8] was able to obtain the best result known, which gives some weak form of the $\mathbb{Z} - RG$–decomposition:

Theorem (Robinson). *If R is a commutative ring, G a nilpotent group, W the augmentation ideal of the group ring RG and A a noetherian RG–module, then the lower RG–central series $\{A_\alpha \mid A_\alpha = AW^\alpha\}$ is stabilized at the first infinite ordinal ω and there is some positive integer n such that $AW^n \cap \zeta_{RG}^{\infty}(A) = \langle 0 \rangle$.*

In this direction, the following result of Zaitsev [12] is worth to mentioning.

Theorem 10. *Let A be a noetherian $\mathbb{Z}G$–module, where G is a hypercentral group. Then A has a non-zero $\mathbb{Z}G$–central image if and only if it has a non-zero $\mathbb{Z}G$–central factor.*

Zaitsev [13] also showed a similar result holds for $\mathfrak{F}$, a result that was deepened by Duan [2].

Theorem 11. *Let A be a noetherian $\mathbb{Z}G$–module, where G is a hyperfinite group. Then A has a non-zero finite $\mathbb{Z}G$–central image if and only if it has a non-zero finite ZG–central factor.*

Theorem 12. *Let A be a noetherian $\mathbb{Z}G$–module, where G is a locally soluble hyperfinite group. Then $A = B \oplus C$, where B is a $\mathbb{Z}G$–module with finite chief factors and C is a $\mathbb{Z}G$–module with infinite chief factors.*

This last result raises the following questions.

Question 1. *Let A be a noetherian $\mathbb{Z}G$–module, where G is a locally soluble hyperfinite group. Find out formations $\mathfrak{X}$ (finite or infinite) for which A has the $\mathfrak{X}$–decomposition.*

Question 2. *Let A be a noetherian $\mathbb{Z}G$–module, where G is a locally soluble FC–hypercentral group. Does have A the $\mathfrak{F}$–decomposition?*

References

[1] Z. Y. Duan. *Extensions of abelian–by–hyper (cyclic or finite) groups.* Comm. Algebra **20** (1992), 2305-2321.

[2] Z. Y. Duan. *The decompositions of noetherian modules over hyperfinite groups.* Ricerche Math. **44** (1995), 65-89.

[3] B. Hartley and M. J. Tomkinson. *Splitting over nilpotent and hyperfinite residuals.* Math. Proc. Cambridge Phil. Soc. **78** (1975), 215-226.

[4] L. A. Kurdachenko, J. Otal and I. Ya. Subbotin. *Baer-like decompositions of modules.* Illinois Math. J., to appear.

[5] L. A. Kurdachenko, B. V. Petrenko and I. Ya. Subbotin. *On generalized hypercenters in artinian modules.* Comm. Algebra **25** (1997), 1023-1046.

[6] L. A. Kurdachenko, B. V. Petrenko and I. Ya. Subbotin. *Direct decompositions in artinian modules over FC–hypercentral groups.* Mat. Contemporanea **14** (1998), 89-99.

[7] B. V. Petrenko. *One generalization of Fitting's theorem defined by the formations of finite groups.* In *The classes of groups with restrictions for subgroups.* Math. Inst. Kyiv 1997, 73-78.

[8] D. J. S. Robinson. *Hypercentral ideals, noetherian modules and theorem of Stroud.* J. Algebra **32** (1974), 234-239.

[9] D. I. Zaitsev. *On the existence of direct complements in groups with operators.* In *Investigations in Group Theory.* Math. Inst. Kyiv, Kyiv 1976, 26-44.

[10] D. I. Zaitsev. *The splitting extensions of abelian groups.* In *The structure of groups and the properties of their subgroups.* Math. Inst. Kyiv 1986, pp. 22-31.

[11] D. I. Zaitsev. *On direct decompositions of abelian groups with operators.* Ukrain. Math. J. **40** (1988), 303-309.

[12] D. I. Zaitsev. *On extensions of abelian groups.* In *The constructive description of groups with prescribed properties of subgroups.* Math. Inst. Kyiv 1980,pp. 16-40.

[13] D. I. Zaitsev. *The hyperfinite extensions of abelian groups.* In *The investigations of groups with restrictions on subgroups*, Math. Inst. Kyiv, Kyiv 1988, 17-26.

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