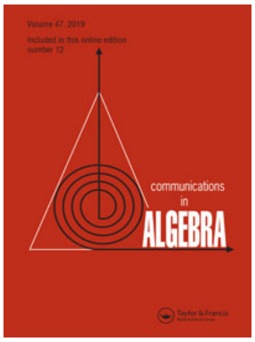

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Certain congruences on free trioids

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ABSTRACT
Loday and Ronco introduced the notion of a trioid and constructed the free trioid of rank 1. This paper is devoted to the study of congruences on trioids. We characterize the least dimonoid congruences and the least semigroup congruence on the free (commutative, rectangular) trioid.

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1. Introduction

Loday and Ronco introduced the notion of a trioid [9]. A triold is a nonempty set T equipped with three binary associative operations $\dashv, \vdash$, and $\perp$ satisfying the following axioms:

$$(x \dashv y) \dashv z = x \dashv (y \vdash z), \tag{T1}$$
$$(x \vdash y) \dashv z = x \vdash (y \dashv z), \tag{T2}$$
$$(x \dashv y) \vdash z = x \vdash (y \vdash z), \tag{T3}$$
$$(x \dashv y) \dashv z = x \dashv (y \perp z), \tag{T4}$$
$$(x \perp y) \dashv z = x \perp (y \dashv z), \tag{T5}$$
$$(x \dashv y) \perp z = x \perp (y \vdash z), \tag{T6}$$
$$(x \vdash y) \perp z = x \vdash (y \perp z), \tag{T7}$$
$$(x \perp y) \vdash z = x \vdash (y \vdash z) \tag{T8}$$

for all $x, y, z \in T$. Originally, trioids arose in algebraic topology. This notion has applications in trialgebra theory [1, 3–5, 9, 11] and lately it was actively studied. Trioid theory has connections to dialgebra theory [2, 6–8, 10] and dimonoid theory [12, 14, 16, 17, 20]. The system of axioms of a trioid includes some axioms of a doppelsemigroup [15, 19]. Some examples of trioids can be found in Section 2 and [18, 21, 22]. For further details, see [9, 13, 18].

The variety theory of trioids was developed in [9, 13, 18, 21, 22]. In these works, free trioids of rank 1 [9], free trioids of an arbitrary rank [18], free n-nilpotent trioids [21], free rectangular

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tribands [22] and free commutative trioids [13] were constructed. For some mentioned free algebras, certain least congruences were characterized. Endomorphisms of free trioids were investigated in [23, 24].

The current paper continues researches from [13, 18, 21, 22]. Its purpose is to study some least congruences on free trioids.

In Section 2, new classes of trioids are constructed.

In Section 3, we characterize the least dimonoid congruences and the least semigroup congruence on the free commutative trioid (Theorem 3.3).

Descriptions of the least dimonoid congruences and the least semigroup congruence on the free trioid (Theorem 4.1) are the topics of Section 4.

In the final section, the least dimonoid congruences and the least semigroup congruence on the free rectangular triband are presented (Theorem 5.3).

2. Classes of trioids

In this section, we give new classes of trioids.

Let X be an alphabet and $F[X]$ the free semigroup on X. For all $h=(p,q)\in X\times X$, assume $[h]=pq\in F[X]$. For every $*\in\{\dashv,\vdash,\perp\}$ let

$$\gamma_*: X\times X\to (X\times X)\cup\{a_1a_2\in F[X]|a_1,a_2\in X\}:(p,q)\mapsto(p,q)\gamma_*$$

be an arbitrary map such that $(p,q)\gamma_*=pq\in F[X]$ or $(p,q)\gamma_*=(p,q)\in X\times X$. Define operations $\dashv,\vdash$, and $\perp$ on $F[X]\cup(X\times X)$ by

$$a_1...a_m*b_1...b_k=a_1...a_mb_1...b_k,$$
$$w*h=w[h],\quad h*w=[h]w,\quad h*f=[h][f],$$
$$p\dashv q=(p,q)\gamma_\dashv,\quad p\vdash q=(p,q)\gamma_\vdash,\quad p\perp q=(p,q)\gamma_\perp$$

for all $a_1...a_m,b_1...b_k\in F[X]$ such that $mk>1$, $w\in F[X],h,f\in X\times X,p,q\in X$ and $*\in\{\dashv,\vdash,\perp\}$. The algebra $(F[X]\cup(X\times X),\dashv,\vdash,\perp)$ will be denoted by $X[\gamma_*]$.

Proposition 2.1. *$X[\gamma_*]$ is a trioid.*

Proof. An immediate verification shows that the axioms of a trioid hold concerning operations $\dashv,\vdash,\perp$ and thus, $X[\gamma_*]$ is a trioid. □

Recall that a dimonoid [8] is a nonempty set T equipped with two binary associative operations $\dashv$ and $\vdash$ satisfying the axioms $(T1)$–$(T3)$. Note that trioids, dimonoids and semigroups are naturally related: if two operations $\dashv$ and $\perp$ or $\vdash$ and $\perp$ of a trioid coincide, we obtain the notion of a dimonoid; if all operations of a trioid coincide, we obtain the notion of a semigroup. For extensive information on dimonoids, see [12, 17, 20].

Remark 2.2. *(i) If $\gamma_\dashv,\gamma_\vdash$, and $\gamma_\perp$ are pairwise distinct, then operations of $X[\gamma_*]$ are pairwise distinct.*

(ii) If $\gamma_\dashv=\gamma_\vdash=\gamma_\perp$, then operations of $X[\gamma_*]$ coincide and it is a semigroup.
(iii) If $\gamma_\vdash=\gamma_\perp$, then operations $\vdash$ and $\perp$ of $X[\gamma_*]$ coincide and it is a dimonoid.
(iv) If $\gamma_\dashv=\gamma_\perp$, then operations $\dashv$ and $\perp$ of $X[\gamma_*]$ coincide and it is a dimonoid.
(v) If $\gamma_\dashv=\gamma_\vdash$, then operations $\dashv$ and $\vdash$ of $X[\gamma_*]$ coincide and it is a dimonoid.

A semigroup S is a rectangular band if $xyx=x$ for all $x,y\in S$. A trioid $(T,\dashv,\vdash,\perp)$ is called a rectangular trioid or a rectangular triband [22] if $(T,\dashv),(T,\vdash)$ and $(T,\perp)$ are rectangular bands. A dimonoid $(D,\dashv,\vdash)$ is called a rectangular dimonoid or a rectangular diband [17] if both semigroups $(D,\dashv)$ and $(D,\vdash)$ are rectangular bands. Free rectangular dimonoids were given in [12, 17].

As usual, $\mathbb{N}$ denotes the set of all positive integers. Let $I_n = \{1, 2, ..., n\}$, $n > 1$, and let $\{X_i\}_{i\in I_n}$ be a family of arbitrary nonempty sets X_i, $i \in I_n$. Define operations $\dashv, \vdash$, and $\perp$ on $\prod_{i\in I_{2k+1}} X_i$, where $k \in \mathbb{N}$, by

$$(x_1, x_2, ..., x_{2k+1}) \dashv (y_1, y_2, ..., y_{2k+1}) = (x_1, x_2, ..., x_{2k}, y_{2k+1}),$$
$$(x_1, x_2, ..., x_{2k+1}) \vdash (y_1, y_2, ..., y_{2k+1}) = (x_1, y_2, ..., y_{2k+1}),$$
$$(x_1, x_2, ..., x_{2k+1}) \perp (y_1, y_2, ..., y_{2k+1}) = (x_1, x_2, ..., x_k, y_{k+1}, ..., y_{2k+1})$$

for all $(x_1, x_2, ..., x_{2k+1}), (y_1, y_2, ..., y_{2k+1}) \in \prod_{i\in I_{2k+1}} X_i$.

Proposition 2.3. *For any $k \in \mathbb{N}$, $(\prod_{i\in I_{2k+1}} X_i, \dashv, \vdash, \perp)$ is a rectangular trioid.*
Proof. The proof is the same as the proof of Lemma 4 [22]. □

Remark 2.4. *Operations $\vdash$ and $\perp$ of a rectangular trioid $(\prod_{i\in I_3} X_i, \dashv, \vdash, \perp)$ coincide and it is a rectangular dimonoid. If $X_i = X$ for all $i \in \{1, 2, 3\}$, then operations $\vdash$ and $\perp$ of $(\prod_{i\in I_3} X_i, \dashv, \vdash, \perp)$ coincide and it is the free rectangular dimonoid.*

Note that in [22] a similar trioid was constructed on the set $\prod_{i\in I_{2k}} X_i, k \in \mathbb{N}$.

3. The least dimonoid congruences on the free commutative trioid

In this section, we characterize the least dimonoid congruences and the least semigroup congruence on the free commutative trioid.

Recall the construction of the free commutative trioid [13].

Let X be an arbitrary nonempty set, $F^\star[X]$ the free commutative semigroup on X, Ω the free monoid on the three-element set $\{a, b, c\}$, and $\theta \in \Omega$ the empty word. Let further ω be an arbitrary word over the alphabet X. The length of ω is denoted by ℓ_ω. By definition, the length ℓ_θ of θ is equal to 0 and $u^0 = \theta$ for any $u \in \Omega\backslash\{\theta\}$. For all $u_1, u_2 \in \Omega$ let

$$f_\dashv(u_1, u_2) = a, \quad f_\vdash(u_1, u_2) = \begin{cases} b, & u_1 = u_2 = \theta, \\ a & \textit{otherwise}, \end{cases}$$

$$f_\perp(u_1, u_2) = \begin{cases} c, & u_1 = c^k, \ u_2 = c^p, \ k, \ p \in \mathbb{N} \cup \{0\}, \\ a & \textit{otherwise}. \end{cases}$$

By $\bar{\Omega}$ denote the subset

$$\left\{ y^k | y \in \{a, c\}, \ k \in \mathbb{N} \cup \{0\} \right\} \cup \{b\}$$

of Ω. Define operations $\dashv, \vdash$, and $\perp$ on

$$A = \left\{ (w, u) \in F^\star[X] \times \bar{\Omega} | \ell_w - \ell_u = 1 \right\}$$

by

$$(w_1, u_1) * (w_2, u_2) = \left(w_1 w_2, f_*(u_1, u_2)^{\ell_{u_1} + \ell_{u_2} + 1} \right)$$

for all $(w_1, u_1), (w_2, u_2) \in A$ and $* \in \{\dashv, \vdash, \perp\}$. The algebra $(A, \dashv, \vdash, \perp)$ is denoted by $FCT(X)$.

Theorem 3.1. ([13], Theorem 3.8) *FCT(X) is the free commutative trioid.*

Recall the construction of the free commutative dimonoid from [12, 17].

Let G be the set of all unordered pairs $(p, q), p, q \in X$. Define operations $\dashv$ and $\vdash$ on $F^\star[X] \cup G$ by

$$a_1 \ldots a_m \dashv b_1 \ldots b_n = a_1 \ldots a_m b_1 \ldots b_n,$$

$$a_1 \ldots a_m \vdash b_1 \ldots b_n = \begin{cases} a_1 \ldots a_m b_1 \ldots b_n, mn > 1, \\ (a_1, b_1), m = n = 1, \end{cases}$$

$$a_1 \ldots a_m \dashv (p,q) = a_1 \ldots a_m \vdash (p,q) = a_1 \ldots a_m pq,$$

$$(p,q) \dashv a_1 \ldots a_m = (p,q) \vdash a_1 \ldots a_m = pqa_1 \ldots a_m,$$

$$(p,q) \dashv (r,s) = (p,q) \vdash (r,s) = pqrs$$

for all $a_1 \ldots a_m, b_1 \ldots b_n \in F^\star[X], (p,q), (r,s) \in G$.

Theorem 3.2. ([17], Theorem 4.1) *$(F^\star[X] \cup G, \dashv, \vdash)$ is the free commutative dimonoid on X.*

The dimonoid $(F^\star[X] \cup G, \dashv, \vdash)$ is denoted by *FCD(X)*.

If $T = (T, \dashv, \vdash)$ is a dimonoid, then the trioid $(T, \dashv, \vdash, \dashv)$ (respectively, $(T, \dashv, \vdash, \vdash)$) is denoted by $(T)^{\dashv}$ (respectively, $(T)^{\vdash}$). Clearly, $(T)^{\dashv}$ and $(T)^{\vdash}$ are distinct as trioids but they coincide as dimonoids. If ρ is a congruence on a trioid $(T, \dashv, \vdash, \perp)$ such that two operations of $(T, \dashv, \vdash, \perp)/\rho$ coincide and it is a dimonoid, we say that ρ is a dimonoid congruence [13]. A dimonoid congruence ρ on a trioid $(T, \dashv, \vdash, \perp)$ is called a $d^{\perp}_{\dashv}$-congruence (respectively, $d^{\perp}_{\vdash}$-congruence) [13] if operations $\dashv$ and $\perp$ (respectively, $\vdash$ and $\perp$) of $(T, \dashv, \vdash, \perp)/\rho$ coincide. If ρ is a congruence on a trioid $(T, \dashv, \vdash, \perp)$ such that all operations of $(T, \dashv, \vdash, \perp)/\rho$ coincide, we say that ρ is a semigroup congruence. If $f : T_1 \to T_2$ is a homomorphism of trioids, the corresponding congruence on T_1 will be denoted by Δ_f.

Now we can formulate the main result of this section.

Theorem 3.3. *Let FCT(X) be the free commutative trioid, $(w_1, u_1), (w_2, u_2) \in FCT(X)$, and b, c as above. Let FCD(X) be the free commutative dimonoid and $F^\star[X]$ the free commutative semigroup.*

(i) *Define a relation $\widetilde{\pi^{\perp}_{\dashv}}$ on FCT(X) by*

$$(w_1, u_1)\widetilde{\pi^{\perp}_{\dashv}}(w_2, u_2)$$

if and only if one of the following conditions holds:

(1) $w_1 = w_2$ *and* $u_1 = u_2 = b$;

(2) $w_1 = w_2$ *and* $u_1 \neq b, \quad u_2 \neq b$.

Then $\widetilde{\pi^{\perp}_{\dashv}}$ is the least $d^{\perp}_{\dashv}$-congruence on FCT(X).

(ii) *Define a relation $\widetilde{\pi^{\perp}_{\vdash}}$ on FCT(X) by*

$$(w_1, u_1)\widetilde{\pi^{\perp}_{\vdash}}(w_2, u_2)$$

if and only if one of the following conditions holds:

(1) $w_1 = w_2$ *and* $u_1, u_2 \in \{b, c\}$;

(2) $w_1 = w_2$ *and* $u_1, u_2 \notin \{b, c\}$.

Then $\widetilde{\pi^{\perp}_{\vdash}}$ is the least $d^{\perp}_{\vdash}$-congruence on FCT(X).

(iii) *Define a relation $\widetilde{\pi}$ on FCT(X) by*

$$(w_1, u_1)\widetilde{\pi}(w_2, u_2)$$

if and only if $w_1 = w_2$. Then $\widetilde{\pi}$ is the least semigroup congruence on FCT(X).

Proof. Let $(x_1 \ldots x_s, y^{s-1}), (z_1 \ldots z_k, t^{k-1}) \in FCT(X)$, where $x_i \in X$ for $1 \leq i \leq s, z_j \in X$ for $1 \leq j \leq k$ and $y, t \in \{a, b, c\}$.

(i) Define a map $\pi^{\perp}_{\dashv} : FCT(X) \to (FCD(X))^{\dashv}$ by

$$\left(x_1 \dots x_s, y^{s-1}\right) \mapsto \left(x_1 \dots x_s, y^{s-1}\right)\pi^{\perp}_{\dashv} = \begin{cases} (x_1, x_2), & s = 2, \quad y = b, \\ x_1 \dots x_s & \text{otherwise.} \end{cases}$$

Let us show that $\pi^{\perp}_{\dashv}$ is an epimorphism. Consider the following four cases:

$$y^{s-1} = t^{k-1} = b, \tag{3.1}$$

$$y^{s-1} \neq b, \quad t^{k-1} \neq b, \tag{3.2}$$

$$y^{s-1} = b, \quad t^{k-1} \neq b, \tag{3.3}$$

$$y^{s-1} \neq b, \quad t^{k-1} = b. \tag{3.4}$$

In the case (3.1), we obtain

$$\begin{aligned} &((x_1x_2, b) * (z_1z_2, b))\pi^{\perp}_{\dashv} = \left(x_1x_2z_1z_2, a^3\right)\pi^{\perp}_{\dashv} \\ &= x_1x_2z_1z_2 = (x_1, x_2) * (z_1, z_2) \\ &= (x_1x_2, b)\pi^{\perp}_{\dashv} * (z_1z_2, b)\pi^{\perp}_{\dashv} \end{aligned}$$

for all $* \in \{\dashv, \vdash, \perp\}$.

If (3.2) holds, then

$$\begin{aligned} &\left(\left(x_1 \dots x_s, y^{s-1}\right) \dashv \left(z_1 \dots z_k, t^{k-1}\right)\right)\pi^{\perp}_{\dashv} = \left(x_1 \dots x_s z_1 \dots z_k, a^{s+k-1}\right)\pi^{\perp}_{\dashv} \\ &= x_1 \dots x_s z_1 \dots z_k = (x_1 \dots x_s) \dashv (z_1 \dots z_k) \\ &= \left(x_1 \dots x_s, y^{s-1}\right)\pi^{\perp}_{\dashv} \dashv \left(z_1 \dots z_k, t^{k-1}\right)\pi^{\perp}_{\dashv}, \end{aligned} \tag{3.5}$$

$$\begin{aligned} &\left(\left(x_1 \dots x_s, y^{s-1}\right) \vdash \left(z_1 \dots z_k, t^{k-1}\right)\right)\pi^{\perp}_{\dashv} \\ &= \left(\begin{cases} (x_1z_1, b), & s = k = 1, \\ \left(x_1 \dots x_s z_1 \dots z_k, a^{s+k-1}\right) & \text{otherwise} \end{cases}\right)\pi^{\perp}_{\dashv} \\ &= \begin{cases} (x_1, z_1), & s = k = 1, \\ x_1 \dots x_s z_1 \dots z_k & \text{otherwise} \end{cases} \\ &= (x_1 \dots x_s) \vdash (z_1 \dots z_k) \\ &= \left(x_1 \dots x_s, y^{s-1}\right)\pi^{\perp}_{\dashv} \vdash \left(z_1 \dots z_k, t^{k-1}\right)\pi^{\perp}_{\dashv}, \\ &\left(\left(x_1 \dots x_s, y^{s-1}\right) \perp \left(z_1 \dots z_k, t^{k-1}\right)\right)\pi^{\perp}_{\dashv} \\ &= \left(\begin{cases} \left(x_1 \dots x_s z_1 \dots z_k, c^{s+k-1}\right), & y = t = c, \\ \left(x_1 \dots x_s z_1 \dots z_k, a^{s+k-1}\right) & \text{otherwise} \end{cases}\right)\pi^{\perp}_{\dashv} \\ &= x_1 \dots x_s z_1 \dots z_k = (x_1 \dots x_s) \dashv (z_1 \dots z_k) \\ &= \left(x_1 \dots x_s, y^{s-1}\right)\pi^{\perp}_{\dashv} \dashv \left(z_1 \dots z_k, t^{k-1}\right)\pi^{\perp}_{\dashv}. \end{aligned} \tag{3.6}$$

Let the case (3.3) is satisfied. Then

$$\begin{aligned} &\left((x_1x_2, b) * \left(z_1 \dots z_k, t^{k-1}\right)\right)\pi^{\perp}_{\dashv} = \left(x_1x_2z_1 \dots z_k, a^{k+1}\right)\pi^{\perp}_{\dashv} \\ &= x_1x_2z_1 \dots z_k = (x_1, x_2) * (z_1 \dots z_k) \\ &= (x_1x_2, b)\pi^{\perp}_{\dashv} * \left(z_1 \dots z_k, t^{k-1}\right)\pi^{\perp}_{\dashv} \end{aligned}$$

for all $* \in \{\dashv, \vdash, \perp\}$.

Further, consider the case (3.4). We obtain

$$\begin{aligned}&\left((x_1 \dots x_s, y^{s-1}) * (z_1 z_2, b)\right)\pi_{\dashv}^{\perp} = (x_1 \dots x_s z_1 z_2, a^{s+1})\pi_{\dashv}^{\perp}\\&= x_1 \dots x_s z_1 z_2 = (x_1 \dots x_s) * (z_1, z_2)\\&= (x_1 \dots x_s, y^{s-1})\pi_{\dashv}^{\perp} * (z_1 z_2, b)\pi_{\dashv}^{\perp}\end{aligned}$$

for all $* \in \{\dashv, \vdash, \perp\}$.

Thus,

$$\left((x_1 \dots x_s, y^{s-1}) * (z_1 \dots z_k, t^{k-1})\right)\pi_{\dashv}^{\perp} = (x_1 \dots x_s, y^{s-1})\pi_{\dashv}^{\perp} * (z_1 \dots z_k, t^{k-1})\pi_{\dashv}^{\perp}$$

for all $(x_1 \dots x_s, y^{s-1}), (z_1 \dots z_k, t^{k-1}) \in FCT(X)$ and $* \in \{\dashv, \vdash, \perp\}$. It is clear that $\pi_{\dashv}^{\perp}$ is a surjection. Since by Theorem 3.2 $FCD(X)$ is the free commutative dimonoid, $(FCD(X))^{\dashv}$ is the trioid which is free in the variety of commutative trioids with $\dashv = \perp$. Consequently, $\Delta_{\pi_{\dashv}^{\perp}}$ is the least $d_{\dashv}^{\perp}$-congruence on $FCT(X)$. From the construction of $\pi_{\dashv}^{\perp}$ it follows that $\Delta_{\pi_{\dashv}^{\perp}} = \widetilde{\pi_{\dashv}^{\perp}}$.

(ii) Define a map $\pi_{\vdash}^{\perp} : FCT(X) \to (FCD(X))^{\vdash}$ by

$$(x_1 \dots x_s, y^{s-1}) \mapsto (x_1 \dots x_s, y^{s-1})\pi_{\vdash}^{\perp} = \begin{cases} (x_1, x_2), & s = 2, \quad y \in \{b, c\},\\ x_1 \dots x_s & \text{otherwise.}\end{cases}$$

We wish to show that $\pi_{\vdash}^{\perp}$ is an epimorphism. There are exist the following four cases:

$$y^{s-1}, \quad t^{k-1} \in \{b, c\}, \tag{3.7}$$

$$y^{s-1}, \quad t^{k-1} \notin \{b, c\}, \tag{3.8}$$

$$y^{s-1} \in \{b, c\}, \quad t^{k-1} \notin \{b, c\}, \tag{3.9}$$

$$y^{s-1} \notin \{b, c\}, \quad t^{k-1} \in \{b, c\}. \tag{3.10}$$

Assume that (3.7) holds. Obviously, (3.7) implies $s = k = 2$ and $y, t \in \{b, c\}$. We get

$$\begin{aligned}&((x_1 x_2, y) * (z_1 z_2, t))\pi_{\vdash}^{\perp} = (x_1 x_2 z_1 z_2, a^3)\pi_{\vdash}^{\perp}\\&= x_1 x_2 z_1 z_2 = (x_1, x_2) * (z_1, z_2)\\&= (x_1 x_2, y)\pi_{\vdash}^{\perp} * (z_1 z_2, t)\pi_{\vdash}^{\perp}\end{aligned}$$

for $* \in \{\dashv, \vdash\}$,

$$\begin{aligned}&((x_1 x_2, y) \perp (z_1 z_2, t))\pi_{\vdash}^{\perp}\\&= \left(\begin{cases} (x_1 x_2 z_1 z_2, c^3), & y = t = c,\\ (x_1 x_2 z_1 z_2, a^3) & \text{otherwise}\end{cases}\right)\pi_{\vdash}^{\perp}\\&= x_1 x_2 z_1 z_2 = (x_1, x_2) \vdash (z_1, z_2)\\&= (x_1 x_2, y)\pi_{\vdash}^{\perp} \vdash (z_1 z_2, t)\pi_{\vdash}^{\perp}.\end{aligned}$$

Let us turn to the case (3.8). We have

$$\begin{aligned}&\left((x_1 \dots x_s, y^{s-1}) \perp (z_1 \dots z_k, t^{k-1})\right)\pi_{\vdash}^{\perp}\\&= \left(\begin{cases} (x_1 z_1, c), & s = k = 1,\\ (x_1 \dots x_s z_1 \dots z_k, a^{s+k-1}) & \text{otherwise}\end{cases}\right)\pi_{\vdash}^{\perp}\\&= \begin{cases} (x_1, z_1), & s = k = 1,\\ x_1 \dots x_s z_1 \dots z_k & \text{otherwise}\end{cases}\\&= (x_1 \dots x_s) \vdash (z_1 \dots z_k)\\&= (x_1 \dots x_s, y^{s-1})\pi_{\vdash}^{\perp} \vdash (z_1 \dots z_k, t^{k-1})\pi_{\vdash}^{\perp}.\end{aligned}$$

The subcases for operations $\dashv$ and $\vdash$ are considered as (3.5) and (3.6), respectively.

Finally, similarly to the cases (3.3) and (3.4), the cases (3.9) and (3.10) are checked.

Consequently, $\pi_{\vdash}^{\perp}$ is a homomorphism. Obviously, $\pi_{\vdash}^{\perp}$ is a surjection. Since by Theorem 3.2 $FCD(X)$ is the free commutative dimonoid, $(FCD(X))^{\vdash}$ is the trioid which is free in the variety of commutative trioids with $\vdash = \perp$. Thus, $\Delta_{\pi_{\vdash}^{\perp}}$ is the least $d_{\vdash}^{\perp}$-congruence on $FCT(X)$. From the definition of $\pi_{\vdash}^{\perp}$ it follows that $\Delta_{\pi_{\vdash}^{\perp}} = \widetilde{\pi_{\vdash}^{\perp}}$.

(iii) Define a map $\pi : FCT(X) \to F^{\star}[X]$ by

$$(x_1 \dots x_s, y^{s-1}) \mapsto (x_1 \dots x_s, y^{s-1})\pi = x_1 \dots x_s.$$

We get

$$\begin{aligned} ((w_1, u_1) * (w_2, u_2))\pi &= \left(w_1 w_2, f_*(u_1, u_2)^{\ell_{u_1} + \ell_{u_2} + 1}\right)\pi \\ &= w_1 w_2 = (w_1, u_1)\pi(w_2, u_2)\pi \end{aligned}$$

for all $(w_1, u_1), (w_2, u_2) \in FCT(X)$ and $* \in \{\dashv, \vdash, \perp\}$.

It means that π is a surjective homomorphism. Since $F^{\star}[X]$ is the free commutative semigroup, Δ_π is the least semigroup congruence on $FCT(X)$. By definition of π, the equality $\Delta_\pi = \widetilde{\pi}$ holds. □

4. The least dimonoid congruences on the free trioid

In this section, we characterize the least dimonoid congruences and the least semigroup congruence on the free trioid. We will use the definitions and notations from Sections 2 and 3.

In [9], Loday and Ronco described the free trioid of rank 1. The free trioid of an arbitrary rank was presented in [18]. Recall this construction.

For any $n, k \in \mathbb{N}$ and $L \subseteq \{1, 2, \dots, n\}$, $L \neq \emptyset$, we let $L + k = \{m + k | m \in L\}$ and denote the least (greatest) number of L by $L_{\min}$ ($L_{\max}$).

Let X be an arbitrary nonempty set. Define operations $\dashv, \vdash$, and $\perp$ on

$$F = \{(w, L) | w \in F[X],\ L \subseteq \{1, 2, \dots, \ell_w\},\ L \neq \emptyset\}$$

by

$$\begin{aligned} (w, L) \dashv (\omega, R) = (w\omega, L), \quad (w, L) \vdash (\omega, R) = (w\omega, R + \ell_w), \\ (w, L) \perp (\omega, R) = (w\omega, L \cup (R + \ell_w)) \end{aligned}$$

for all $(w, L), (\omega, R) \in F$. By Lemma 7.1 and Theorem 7.1 from [18], $(F, \dashv, \vdash, \perp)$ is the free trioid. It is denoted by $FT(X)$.

The free dimonoid was first constructed in [8]. A dimonoid which is isomorphic to the free dimonoid can be found in [17]. Recall this construction.

Define operations $\dashv$ and $\vdash$ on

$$\mathbb{F} = \left\{(w, m) \in F[X] \times \mathbb{N} | \ell_w \geq m\right\}$$

by

$$\begin{aligned} (w_1, m_1) \dashv (w_2, m_2) &= (w_1 w_2, m_1), \\ (w_1, m_1) \vdash (w_2, m_2) &= (w_1 w_2, \ell_{w_1} + m_2) \end{aligned}$$

for all $(w_1, m_1), (w_2, m_2) \in \mathbb{F}$. The algebra $(\mathbb{F}, \dashv, \vdash)$ is denoted by $\breve{F}[X]$. By Lemmas 3.2 and 3.3 from [17], $\breve{F}[X]$ is the free dimonoid.

The following theorem characterizes the least dimonoid congruences and the least semigroup congruence on $FT(X)$.

Theorem 4.1. *Let FT(X) be the free trioid,* $(w, L), (\omega, R) \in FT(X)$*. Let* $\breve{F}[X]$ *be the free dimonoid and* $F[X]$ *the free semigroup.*

(i) Define a relation $\widetilde{\sigma_{\dashv}^{\perp}}$ *on FT(X) by*

$$(w, L)\widetilde{\sigma_{\dashv}^{\perp}}(\omega, R)$$

if and only if

$$w = \omega \quad \text{and} \quad L_{min} = R_{min}.$$

Then $\widetilde{\sigma_{\dashv}^{\perp}}$ *is the least* $d_{\dashv}^{\perp}$*-congruence on FT(X).*

(ii) Define a relation $\widetilde{\sigma_{\vdash}^{\perp}}$ *on FT(X) by*

$$(w, L)\widetilde{\sigma_{\vdash}^{\perp}}(\omega, R)$$

if and only if

$$w = \omega \quad \text{and} \quad L_{max} = R_{max}.$$

Then $\widetilde{\sigma_{\vdash}^{\perp}}$ *is the least* $d_{\vdash}^{\perp}$*-congruence on FT(X).*

(iii) Define a relation $\widetilde{\sigma}$ *on FT(X) by*

$$(w, L)\widetilde{\sigma}(\omega, R)$$

if and only if $w = \omega$*. Then* $\widetilde{\sigma}$ *is the least semigroup congruence on FT(X).*

Proof. (i) Define a map $\sigma_{\dashv}^{\perp} : FT(X) \to (\breve{F}[X])^{\dashv}$ by

$$(w, L) \mapsto (w, L)\sigma_{\dashv}^{\perp} = (w, L_{min}).$$

We have

$$\begin{aligned}
((w, L) \dashv (\omega, R))\sigma_{\dashv}^{\perp} &= (w\omega, L)\sigma_{\dashv}^{\perp} = (w\omega, L_{min}) \\
&= (w, L_{min}) \dashv (\omega, R_{min}) = (w, L)\sigma_{\dashv}^{\perp} \dashv (\omega, R)\sigma_{\dashv}^{\perp}, \\
((w, L) \vdash (\omega, R))\sigma_{\dashv}^{\perp} &= (w\omega, R + \ell_w)\sigma_{\dashv}^{\perp} = \big(w\omega, (R + \ell_w)_{min}\big) \\
&= (w\omega, R_{min} + \ell_w) = (w, L_{min}) \vdash (\omega, R_{min}) = (w, L)\sigma_{\dashv}^{\perp} \vdash (\omega, R)\sigma_{\dashv}^{\perp}, \\
((w, L)\perp(\omega, R))\sigma_{\dashv}^{\perp} &= (w\omega, L \cup (R + \ell_w))\sigma_{\dashv}^{\perp} = \big(w\omega, (L \cup (R + \ell_w))_{min}\big) \\
&= (w\omega, L_{min}) = (w, L_{min}) \dashv (\omega, R_{min}) = (w, L)\sigma_{\dashv}^{\perp} \dashv (\omega, R)\sigma_{\dashv}^{\perp}.
\end{aligned}$$

Thus, $\sigma_{\dashv}^{\perp}$ is a surjective homomorphism. Since $\breve{F}[X]$ is the free dimonoid, $(\breve{F}[X])^{\dashv}$ is the trioid which is free in the variety of trioids with $\dashv = \perp$. It means that $\Delta_{\sigma_{\dashv}^{\perp}}$ is the least $d_{\dashv}^{\perp}$-congruence on $FT(X)$. From the construction of $\sigma_{\dashv}^{\perp}$ it follows that $\Delta_{\sigma_{\dashv}^{\perp}} = \widetilde{\sigma_{\dashv}^{\perp}}$.

(ii) Define a map $\sigma_{\vdash}^{\perp} : FT(X) \to (\breve{F}[X])^{\vdash}$ by

$$(w, L) \mapsto (w, L)\sigma_{\vdash}^{\perp} = (w, L_{max}).$$

We get

$$\begin{aligned}
((w, L) \dashv (\omega, R))\sigma_{\vdash}^{\perp} &= (w\omega, L)\sigma_{\vdash}^{\perp} = (w\omega, L_{max}) \\
&= (w, L_{max}) \dashv (\omega, R_{max}) = (w, L)\sigma_{\vdash}^{\perp} \dashv (\omega, R)\sigma_{\dashv}^{\perp}, \\
((w, L) \vdash (\omega, R))\sigma_{\vdash}^{\perp} &= (w\omega, R + \ell_w)\sigma_{\vdash}^{\perp} = \big(w\omega, (R + \ell_w)_{max}\big) \\
&= (w\omega, R_{max} + \ell_w) = (w, L_{max}) \vdash (\omega, R_{max}) = (w, L)\sigma_{\vdash}^{\perp} \vdash (\omega, R)\sigma_{\vdash}^{\perp}, \\
((w, L)\perp(\omega, R))\sigma_{\vdash}^{\perp} &= (w\omega, L \cup (R + \ell_w))\sigma_{\vdash}^{\perp} = \big(w\omega, (L \cup (R + \ell_w))_{max}\big) \\
&= (w\omega, R_{max} + \ell_w) = (w, L_{max}) \vdash (\omega, R_{max}) = (w, L)\sigma_{\vdash}^{\perp} \vdash (\omega, R)\sigma_{\vdash}^{\perp}.
\end{aligned}$$

Consequently, $\sigma_{\vdash}^{\perp}$ is a surjective homomorphism. Since $\breve{F}[X]$ is the free dimonoid, $(\breve{F}[X])^{\vdash}$ is the trioid which is free in the variety of trioids with $\vdash = \perp$. Hence, $\Delta_{\sigma_{\vdash}^{\perp}}$ is the least $d_{\vdash}^{\perp}$-congruence on $FT(X)$. From the definition of $\sigma_{\vdash}^{\perp}$ it follows that $\Delta_{\sigma_{\vdash}^{\perp}} = \widetilde{\sigma_{\vdash}^{\perp}}$.

(iii) Define a map $\sigma : FT(X) \to F[X]$ by

$$(w, L) \mapsto (w, L)\sigma = w.$$

It is easy to see that

$$((w, L) * (\omega, R))\sigma = (w\omega, H_*)\sigma = w\omega = (w, L)\sigma(\omega, R)\sigma$$

for some $H_* \subseteq \{1, 2, ..., \ell_{w\omega}\}$ and all $* \in \{\dashv, \vdash, \perp\}$. It means that σ is a surjective homomorphism. Since $F[X]$ is the free semigroup, Δ_σ is the least semigroup congruence on $FT(X)$. By the construction of σ, relations Δ_σ and $\widetilde{\sigma}$ coincide. □

5. The least dimonoid congruences on the free rectangular trioid

In this section, we characterize the least dimonoid congruences and the least semigroup congruence on the free rectangular trioid. We will use the definitions and notations from Sections 2 and 3.

We recall the construction of the free rectangular trioid [22].

Let X be an arbitrary nonempty set and $X^4 = X \times X \times X \times X$. Define operations $\dashv, \vdash$, and $\perp$ on X^4 by

$$(x_1, x_2, x_3, x_4) \dashv (y_1, y_2, y_3, y_4) = (x_1, x_2, x_3, y_4),$$
$$(x_1, x_2, x_3, x_4) \vdash (y_1, y_2, y_3, y_4) = (x_1, y_2, y_3, y_4),$$
$$(x_1, x_2, x_3, x_4) \perp (y_1, y_2, y_3, y_4) = (x_1, x_2, y_3, y_4)$$

for all $(x_1, x_2, x_3, x_4), (y_1, y_2, y_3, y_4) \in X^4$. The algebra $(X^4, \dashv, \vdash, \perp)$ is denoted by $FRT(X)$.

Theorem 5.1. ([22], Theorem 1) *FRT(X) is the free rectangular triband.*

Now we recall the construction of the free rectangular dimonoid [12, 17] (see also Remark 2.4).

Define operations $\dashv$ and $\vdash$ on $X^3 = X \times X \times X$ by

$$(x_1, x_2, x_3) \dashv (y_1, y_2, y_3) = (x_1, x_2, y_3),$$
$$(x_1, x_2, x_3) \vdash (y_1, y_2, y_3) = (x_1, y_2, y_3)$$

for all $(x_1, x_2, x_3), (y_1, y_2, y_3) \in X^3$. The algebra $(X^3, \dashv, \vdash)$ is denoted by $FRct(X)$.

Theorem 5.2. ([17], Theorem 5.1) *FRct(X) is the free rectangular dimonoid.*

Further, consider the construction of the free rectangular band.

Define the multiplication on $X^2 = X \times X$ by

$$(x_1, x_2)(y_1, y_2) = (x_1, y_2)$$

for all $(x_1, x_2), (y_1, y_2) \in X^2$. Denote the obtained algebra by X_{rb}. It is well known that X_{rb} is the free rectangular band.

The main result of this section is the following theorem.

Theorem 5.3. *Let FRT(X) be the free rectangular triband, $(x_1, x_2, x_3, x_4), (y_1, y_2, y_3, y_4) \in FRT(X)$. Let FRct(X) be the free rectangular dimonoid and X_{rb} the free rectangular band.*

(i) *Define a relation* $\widetilde{\psi_{\dashv}^{\perp}}$ *on FRT(X) by*

$$(x_1, x_2, x_3, x_4)\widetilde{\psi_{\dashv}^{\perp}}(y_1, y_2, y_3, y_4)$$

if and only if

$$(x_1, x_2, x_4) = (y_1, y_2, y_4).$$

Then $\widetilde{\psi_{\dashv}^{\perp}}$ *is the least* $d_{\dashv}^{\perp}$*-congruence on FRT(X).*

(ii) *Define a relation* $\widetilde{\psi_{\vdash}^{\perp}}$ *on FRT(X) by*

$$(x_1, x_2, x_3, x_4)\widetilde{\psi_{\vdash}^{\perp}}(y_1, y_2, y_3, y_4)$$

if and only if

$$(x_1, x_3, x_4) = (y_1, y_3, y_4).$$

Then $\widetilde{\psi_{\vdash}^{\perp}}$ *is the least* $d_{\vdash}^{\perp}$*-congruence on FRT(X).*

(iii) *Define a relation* $\widetilde{\psi}$ *on FRT(X) by*

$$(x_1, x_2, x_3, x_4)\widetilde{\psi}(y_1, y_2, y_3, y_4)$$

if and only if

$$(x_1, x_4) = (y_1, y_4).$$

Then $\widetilde{\psi}$ *is the least semigroup congruence on FRT(X).*

Proof. (i) Define a map $\psi_{\dashv}^{\perp} : FRT(X) \to (FRct(X))^{\dashv}$ by

$$(x_1, x_2, x_3, x_4) \mapsto (x_1, x_2, x_3, x_4)\psi_{\dashv}^{\perp} = (x_1, x_2, x_4).$$

By Theorem 4 (iii) from [22], $\psi_{\dashv}^{\perp}$ is a homomorphism. It is obvious that $\psi_{\dashv}^{\perp}$ is a surjective map. Since by Theorem 5.2 $FRct(X)$ is the free rectangular dimonoid, $(FRct(X))^{\dashv}$ is the trioid which is free in the variety of rectangular trioids with $\dashv = \perp$. It means that $\Delta_{\psi_{\dashv}^{\perp}}$ is the least $d_{\dashv}^{\perp}$-congruence on $FRT(X)$. From the construction of $\psi_{\dashv}^{\perp}$ it follows that $\Delta_{\psi_{\dashv}^{\perp}} = \widetilde{\psi_{\dashv}^{\perp}}$.

(ii) Define a map $\psi_{\vdash}^{\perp} : FRT(X) \to (FRct(X))^{\vdash}$ by

$$(x_1, x_2, x_3, x_4) \mapsto (x_1, x_2, x_3, x_4)\psi_{\vdash}^{\perp} = (x_1, x_3, x_4).$$

By Theorem 4 (iv) from [22], $\psi_{\vdash}^{\perp}$ is a homomorphism. It is easy to see that $\psi_{\vdash}^{\perp}$ is a surjection. Since by Theorem 5.2 $FRct(X)$ is the free rectangular dimonoid, $(FRct(X))^{\vdash}$ is the trioid which is free in the variety of rectangular trioids with $\vdash = \perp$. Hence, $\Delta_{\psi_{\vdash}^{\perp}}$ is the least $d_{\vdash}^{\perp}$-congruence on $FRT(X)$. From the definition of $\psi_{\vdash}^{\perp}$ it follows that $\Delta_{\psi_{\vdash}^{\perp}} = \widetilde{\psi_{\vdash}^{\perp}}$.

(iii) Define a map $\psi : FRT(X) \to X_{rb}$ by

$$(x_1, x_2, x_3, x_4) \mapsto (x_1, x_2, x_3, x_4)\psi = (x_1, x_4).$$

According to Theorem 2 (iii) from [22], ψ is a homomorphism. Evidently, ψ is a surjective map. From Corollary 2 (iii) of [22], it follows that Δ_{ψ} is the least semigroup congruence on $FRT(X)$. By definition of ψ, we have $\Delta_{\psi} = \widetilde{\psi}$. □

Since associative trialgebras [9] are linear analogs of trioids, all results obtained for trioids hold for associative trialgebras.

References

[1] Bagherzadeh, F., Bremner, M., Madariaga, S. (2017). Jordan trialgebras and post-Jordan algebras. *J. Algebra* 486:360–395. DOI: 10.1016/j.jalgebra.2017.04.022.

[2] Bokut, L. A., Chen, Y., Liu, C. (2010). Gröbner–Shirshov bases for dialgebras. *Int. J. Algebra Comput.* 20(3): 391–415. DOI: 10.1142/S0218196710005753.

[3] Casas, J. M. (2006). Trialgebras and Leibniz 3-algebras. *Bol. Soc. Mat. Mex.* 12(2):165–178.

[4] Ebrahimi-Fard, K. J. (2002). Loday-type algebras and the Rota-Baxter relation. *Lett. Math. Phys.* 61(2): 139–147.

[5] Gubarev, V., Kolesnikov, P. (2013). Embedding of dendriform algebras into Rota-Baxter algebras. *Cent. Eur. J. Math.* 11(2):226–245. DOI: 10.2478/s11533-012-0138-z.

[6] Kolesnikov, P. S., Voronin, V. Y. (2013). On special identities for dialgebras. *Linear Multilinear Algebra* 61(3):377–391. DOI: 10.1080/03081087.2012.686108.

[7] Kolesnikov, P. S. (2008). Varieties of dialgebras and conformal algebras. *Sib. Math. J.* 49(2):257–272. DOI: 10.1007/s11202-008-0026-8.

[8] Loday, J.-L. (2001). Dialgebras. In: *Dialgebras and Related Operads: Lecture Notes on Mathematics*, Vol. 1763, Berlin: Springer-Verlag, pp. 7–66.

[9] Loday, J.-L., Ronco, M. O. (2004). Trialgebras and families of polytopes. *Contemp. Math.* 346:369–398.

[10] Pozhidaev, A. P. (2008). Dialgebras and related triple systems. *Sib. Math. J.* 49(4):696–708. DOI: 10.1007/s11202-008-0067-z.

[11] Vallette, B. (2007). Homology of generalized partition posets. *J. Pure Appl. Algebra* 208(2):699–725. DOI: 10.1016/j.jpaa.2006.03.012.

[12] Zhuchok, A. V. (2015). Dimonoids and bar-units. *Sib. Math. J.* 56(5):827–840. DOI: 10.1134/S0037446615050055.

[13] Zhuchok, A. V. (2019). Free commutative trioids. *Semigroup Forum* 98(2):355–368. DOI: 10.1007/s00233-019-09995-y.

[14] Zhuchok, A. V. (2013). Free products of dimonoids. *Quasigroups Relat. Syst.* 21(2):273–278.

[15] Zhuchok, A. V. (2017). Free products of doppelsemigroups. *Algebra Univers.* 77(3):361–374. DOI: 10.1007/s00012-017-0431-6.

[16] Zhuchok, A. V. (2011). Semilatties of subdimonoids. *Asian Eur. J. Math.* 4 (2):359–371. DOI: 10.1142/S1793557111000290.

[17] Zhuchok, A. V. (2017). Structure of relatively free dimonoids. *Commun. Algebra* 45(4):1639–1656. DOI: 10.1080/00927872.2016.1222404.

[18] Zhuchok, A. V. (2015). Trioids. Asian Eur. J. Math. 8(4):1550089. DOI: 10.1142/S1793557115500898.

[19] Zhuchok, A. V., Demko, M. (2016). Free n-dinilpotent doppelsemigroups. *Algebra Discrete Math.* 22(2): 304–316.

[20] Zhuchok, A. V., Zhuchok, Yul. V. (2016). Free left n-dinilpotent dimonoids. *Semigroup Forum* 93(1): 161–179. DOI: 10.1007/s00233-015-9743-z.

[21] Zhuchok, Yul. V. (2015). Free n-nilpotent trioids. *Mat. Stud.* 43(1):3–11.

[22] Zhuchok, Yul. V. (2015). Free rectangular tribands. *Bul. Acad. Stiinte Repub. Mold. Mat.* 78(2):61–73.

[23] Zhuchok, Y. V. (2015). On the determinability of free trioids by semigroups of endomorphisms. Dopov. Nac. Akad. Nauk. Ukr. 4:7–11. (in Russian). DOI: 10.15407/dopovidi2015.04.007.

[24] Zhuchok, Y. V. (2016). The endomorphism monoid of a free trioid of rank 1. *Algebra Univers.* 76(3): 355–366. DOI: 10.1007/s00012-016-0392-1.