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On cospectral signed digraphs

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Abstract. The set of distinct eigenvalues of a signed digraph S together with their respective multiplicities is called its spectrum. Two signed digraphs of same order are said to be cospectral if they have the same spectrum. In this paper, we show the existence of integral, real and Gaussian cospectral signed digraphs. We give a spectral characterization of normal signed digraphs and use it to construct cospectral normal signed digraphs.

1. Introduction

A signed digraph is defined to be a pair $S = (D, \sigma)$, where $D = (V, \mathscr{A})$ is the underlying digraph and $\sigma : \mathscr{A} \to \{-1, 1\}$ is the signing function. The sets of positive and negative arcs of S are respectively denoted by $\mathscr{A}^+$ and $\mathscr{A}^-$. So the arc set of S is $\mathscr{A} = \mathscr{A}^+ \cup \mathscr{A}^-$. A signed digraph is said to be homogeneous if all of its arcs have either positive sign or negative sign and heterogeneous, otherwise. Throughout this paper, the bold arcs will denote positive arcs and the dotted arcs will denote negative arcs.

An arc from a vertex u to the vertex v is represented by (u, v). A path of length $n-1$ ($n \geqslant 2$), denoted by P_n, is a signed digraph on n vertices $v_1, v_2, \ldots, v_n$ with $n-1$ signed arcs (v_i, v_{i+1}), $i = 1, 2, \ldots, n-1$. A cycle of length n is a signed digraph having vertices $v_1, v_2, \ldots, v_n$ and signed arcs (v_i, v_{i+1}), $i = 1, 2, \ldots, n-1$ and (v_n, v_1). A signed digraph S is said to be strongly connected if its underlying digraph S^u is strongly

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connected. The sign of a signed digraph is defined as the product of signs of its arcs. A signed digraph is said to be positive (negative) if its sign is positive (negative), i.e., it contains an even (odd) number of negative arcs. A signed digraph is said to be all-positive (respectively, all-negative) if all of its arcs are positive (respectively negative). A signed digraph is said to be cycle-balanced if each of its cycles is positive and non-cycle-balanced, otherwise. We denote the positive and negative cycle of order n respectively by C_n and $\mathbf{C}_n$, where n is the number of vertices.

The adjacency matrix of a signed digraph S with vertex set $\{v_1, \ldots, v_n\}$ is the $n \times n$ matrix $A(S) = (a_{ij})$, where

$$a_{ij} = \begin{cases} \sigma(v_i, v_j) & \text{if there is an arc from } v_i \text{ to } v_j, \\ 0 & \text{otherwise.} \end{cases}$$

The characteristic polynomial $|xI - A(S)|$ of the adjacency matrix $A(S)$ of a signed digraph S is called the characteristic polynomial of S and is denoted by $\phi_S(x)$. The eigenvalues of $A(S)$ are called the eigenvalues of S. As $A(S)$ is not necessarily real symmetric, so eigenvalues can be complex numbers. The set of distinct eigenvalues of S together with their respective multiplicities is called the spectrum of S. If S is a signed digraph of order n with distinct eigenvalues $z_1, z_2, \ldots, z_k$ and if their respective multiplicities are $m_1, m_2, \ldots, m_k$, we write the spectrum of S as $\operatorname{spec}(S) = \{z_1^{(m_1)}, z_2^{(m_2)}, \ldots, z_k^{(m_k)}\}$.

A linear signed subdigraph of a signed digraph S is a signed subdigraph with indegree and outdegree of each vertex equal to one, i.e., each component is a cycle.

The following theorem connects the coefficients of the characteristic polynomial of a signed digraph with its structure [1].

Theorem 1. *If S is a signed digraph with characteristic polynomial*

$$\phi_S(x) = x^n + a_1 x^{n-1} + \cdots + a_{n-1}x + a_n,$$

then

$$a_j = \sum_{L \in \pounds_j} (-1)^{p(L)} \prod_{Z \in c(L)} s(Z),$$

for all $j = 1, 2, \ldots, n$, where $\pounds_j$ is the set of all linear signed subdigraphs L of S of order j, $p(L)$ denotes the number of components of L, $c(L)$ denotes the set of all cycles of L and $s(Z)$ denotes the sign of cycle Z.

The spectral criterion for cycle-balance of signed digraphs given by Acharya [1] is as follows.

Theorem 2. *A signed digraph S is cycle-balanced if and only if it is cospectral with the underlying unsigned digraph.*

Two signed digraphs of the same order are said to be cospectral (or isospectral) if they have the same spectrum and non-cospectral, otherwise. Esser and Harary [6] studied digraphs with integral, real and Gaussian spectra. A signed digraph is said to be normal if its adjacency matrix is normal. In this paper, we show the existence of signed digraphs with integral, real and Gaussian spectra. We give a spectral characterization of normal signed digraphs and as a consequence we construct cospectral normal signed digraphs.

2. Existence of cospectral signed digraphs

Let $S_1 = (V_1, \mathscr{A}_1, \sigma_1)$ and $S_2 = (V_2, \mathscr{A}_2, \sigma_2)$ be two signed digraphs, their Cartesian product (or sum) [9] denoted by $S_1 \times S_2$ is the signed digraph $(V_1 \times V_2, \mathscr{A}, \sigma)$, where the arc set $\mathscr{A}$ is that of the Cartesian product of underlying unsigned digraphs and the sign function is defined by

$$\sigma((u_i, v_j), (u_k, v_l)) = \begin{cases} \sigma_1(u_i, u_k) & \text{if } j = l, \\ \sigma_2(v_j, v_l) & \text{if } i = k. \end{cases}$$

Unlike Kronecker product [8], Cartesian product of two strongly connected signed digraphs is always strongly connected as can be seen in the following result.

Lemma 1. *If S_1 and S_2 are two strongly connected signed digraphs, then $S_1 \times S_2$ is strongly connected.*

Proof. Let $(u_i, v_j), (u_p, v_q) \in V(S_1 \times S_2)$, where we assume $p \leqslant q$ (case $p > q$ can be dealt similarly). Since S_1 is strongly connected, there exists a directed path $(u_i, u_{i+1})(u_{i+1}, u_{i+2}) \dots (u_{p-1}, u_p)$. Also, strong connectedness of S_2 implies the existence of a directed path

$$(v_j, v_{j+1})(v_{j+1}, v_{j+2}) \dots (v_{q-1}, v_q).$$

By definition of Cartesian product, Fig. 1 illustrates the existence of a directed path from (u_i, v_j) to (u_p, v_q). Signs do not play any role in connectedness, so we take all arcs in Fig. 1 to be positive. Similarly, one can prove the reverse part. $\square$

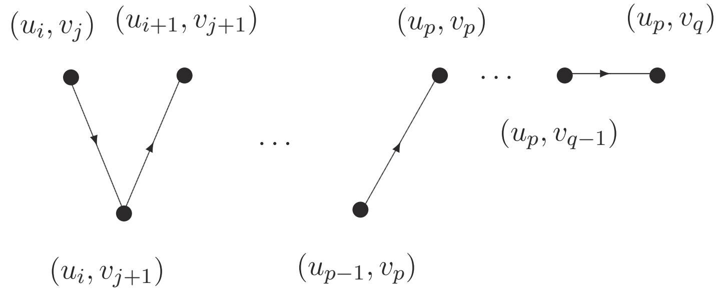


Figure 1. Strong connectedness of Cartesian product of signed digraphs.

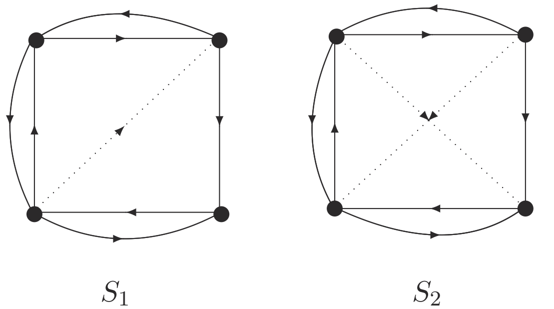


Figure 2. A pair of integral cospectral signed digraphs.

Definition 1. A signed digraph S is said to integral, or real, or Gaussian according as the spectrum of S is integral, or real, or Gaussian respectively.

The following result shows the existence of non-cycle-balanced integral signed digraphs.

Theorem 3. *For each positive integer $n \geqslant 4$, there exists a family of n integral cospectral, strongly connected, non-symmetric and non-cycle-balanced signed digraphs of order 4^n.*

Proof. Consider the signed digraphs S_1 and S_2 shown in Fig. 2. Clearly S_1 and S_2 are non-cycle-balanced and strongly connected. By Theorem 1,

$$\phi_{S_1}(x) = \phi_{S_2}(x) = x^4 - 3x^2 + 2x.$$

Therefore, $\operatorname{spec}(S_1) = \operatorname{spec}(S_2) = \{-2, 0, 1^{(2)}\}$. That is, S_1 and S_2 are integral cospectral. Let

$$S^{(k)} = S_1 \times S_1 \times \cdots \times S_1 \times S_2 \times S_2 \times \cdots \times S_2,$$

where we take k copies of S_1 and $n-k$ copies of S_2. Clearly, for each n, we have n cospectral signed digraphs $S^{(k)}, \quad k = 1, 2, \ldots, n$ of order 4^n. S_1 and S_2 are non-symmetric implies $S^{(k)}$ is non-symmetric. By repeated application of Lemma 1 and using the fact that the Cartesian product of signed digraphs is cycle-balanced if and only if the constituent signed digraphs are cycle-balanced [Theorem 4.8, [9]], the result follows. □

Integral signed digraphs are obviously real. There exist non-integral real signed digraphs as can be see in the following result.

Theorem 4. *For each positive integer $n \geqslant 4$, there exists a family of n real cospectral, strongly connected, non-symmetric and non-cycle-balanced signed digraphs of order 4^n.*

Proof. Consider the signed digraphs S_1, S_2 shown in Fig. 3. Clearly, both signed digraphs are non-cycle-balanced and strongly connected. By Theorem 1,

$$\phi_{S_1}(x) = \phi_{S_2}(x) = x^4 - 3x^2 + 2.$$

Therefore, $\mathrm{spec}(S_1) = \mathrm{spec}(S_2) = \{-\sqrt{2}, -1, 1, \sqrt{2}\}$. Proceed in a similar way as in Theorem 3, the result follows. □

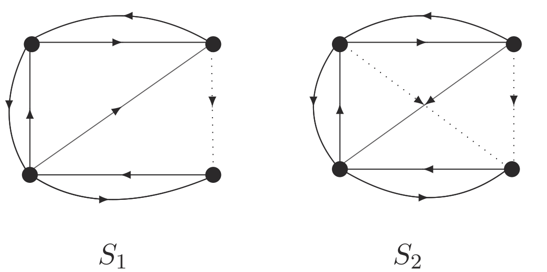
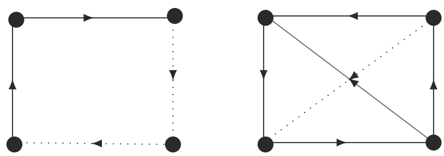


Figure 3. A pair of real cospectral signed digraphs.

Every integral signed digraph is obviously Gaussian. The next result shows that there exist non-integral Gaussian signed digraphs, i.e., signed digraphs with eigenvalues of the form $a + \imath b$, where a and b are integers with $b \neq 0$, for some b.

Theorem 5. *For each positive integer $n \geqslant 4$, there exists a collection of n Gaussian cospectral, strongly connected, non-symmetric and non-cycle-balanced signed digraphs of order 4^n.*

Proof. Consider the signed digraphs S_1 and S_2 as shown in Fig. 4. Clearly S_1 is cycle-balanced whereas S_2 is non-cycle-balanced. Moreover both signed digraphs are strongly connected. By Theorem 1, we have $\phi_{S_1}(x) = \phi_{S_2}(x) = x^4 - 1$. Therefore, $\operatorname{spec}(S_1) = \operatorname{spec}(S_2) = \{-1, 1, -\iota, \iota\}$. Hence S_1 and S_2 are Gaussian cospectral. Proceed in a similar way as in Theorem 3, the result follows. $\square$



S_1 S_2

FIGURE 4. A pair of Gaussian cospectral signed digraphs.

Two digraphs D_1 and D_2 are said to be quasi-cospectral if there exist cospectral signed digraphs S_1 and S_2 on them respectively. Two cospectral digraphs are quasi-cospectral by Theorem 2, as we can take any two cycle-balanced signed digraphs one on each digraph. Two digraphs are said to be strictly quasi-cospectral if they are quasi-cospectral but not cospectral. Two digraphs D_1 and D_2 are said to be strongly quasi-cospectral if both D_1 and D_2 are cospectral and there exist non-cycle-balanced cospectral signed digraphs S_1 and S_2 on them respectively. It is clear that if D_1 and D_2 are strongly quasi-cospectral digraphs, then both should have at least on cycle. For quasi-cospectral and strongly quasi-cospectral graphs and digraphs see [2, 3].

Definition 2. We say two digraphs D_1 and D_2 are integral, real and Gaussian strongly quasi-cospectral if both D_1 and D_2 are respectively integral, real and Gaussian cospectral and there exists non-cycle-balanced signed digraphs S_1 and S_2 on them which are respectively integral, real and Gaussian cospectral.

The following two result show the existence of an integral and real strongly quasi-cospectral digraphs.

Theorem 6. *For each positive integer $n \geqslant 4$, there exists a family of n integral, strongly connected, non-symmetric and strongly quasi-cospectral digraphs of order 4^n.*

Proof. Let D_1 and D_2 respectively be the underlying digraphs of integral signed digraphs S_1 and S_2 shown in Fig. 2. Then D_1 and D_2 are all-positive signed digraphs. By Theorem 1, we have

$$\phi_{D_1}(x) = \phi_{D_2}(x) = x^4 - 3x^2 - 2x.$$

Therefore, $\mathrm{spec}(D_1) = \mathrm{spec}(D_2) = \{-1^{(2)}, 0, 2\}$.

Take $D^{(k)} = D_1 \times D_1 \times \cdots \times D_1 \times D_2 \times D_2 \times \cdots \times D_2$, where we take k copies of D_1 and $n-k$ copies of D_2. In this way, for each $n \geqslant 4$, we get n cospectral non-symmetric and strongly connected integral digraphs. Thus for any two of these integral cospectral digraphs $D^{(k_1)}$ and $D^{(k_2)}$ there exist corresponding non-cycle-balanced signed digraphs $S^{(k_1)}$ and $S^{(k_2)}$ on them which are integral cospectral. □

The following result shows the existence of real strongly quasi cospectral digraphs.

Theorem 7. *For each positive integer $n \geqslant 4$, there exists a collection of n real, strongly connected, non-symmetric and strongly quasi-cospectral digraphs of order 4^n.*

Proof. Let D_1 and D_2 be the underlying digraphs of signed digraphs S_1 and S_2 as shown in Fig. 3. It is easy to see that $\phi_{D_1}(x) = \phi_{D_2}(x) = x^4 - 3x^2 - 2x$ and $\mathrm{spec}(D_1) = \mathrm{spec}(D_2) = \{-1^{(2)}, 0, 2\}$. Also $\mathrm{spec}(S_1) = \mathrm{spec}(S_2) = \{-\sqrt{2}, -1, 1, \sqrt{2}\}$. Thus D_1 and D_2 are real strongly quasi-cospectral. Applying the same technique as in Theorem 6, the result follows. □

3. Normal signed digraphs

We start with the following definition.

Definition 3. A signed digraph S is said to normal if its adjacency matrix $A(S)$ is normal.

The following result can be seen in [9].

Lemma 2. *Let S be a signed digraph having n vertices and a arcs and let $z_1, z_2, \ldots, z_n$ be its eigenvalues. Then*

(i) $\sum_{j=1}^{n}(\Re z_j)^2 - \sum_{j=1}^{n}(\Im z_j)^2 = c_2^+ - c_2^-$,

(ii) $\sum_{j=1}^{n}(\Re z_j)^2 + \sum_{j=1}^{n}(\Im z_j)^2 \leqslant a = a^+ + a^-$, *where c_2^+ and c_2^- are the number of closed positive and negative walks of length 2 of the signed digraph S respectively.*

The following result characterizes the normal signed digraphs in terms of the spectra.

Theorem 8. *Let S be a signed digraph on n vertices, $a = a^+ + a^-$ arcs, c_2^+ closed positive walks of length 2, c_2^- closed negative walks of length 2 and let $z_1, z_2, \ldots, z_n$ be its eigenvalues. Then the following statements are equivalent.*

(i) *S is normal;*
(ii) $\sum_{j=1}^{n} |z_j|^2 = a$;
(iii) $\sum_{j=1}^{n} (\Re z_j)^2 = \frac{1}{2}(a + (c_2^+ - c_2^-))$;
(iv) $\sum_{j=1}^{n} (\Im z_j)^2 = \frac{1}{2}(a - (c_2^+ - c_2^-))$.

Proof. (i) $\Longleftrightarrow$ (ii). From [7], we note that a matrix $A = (a_{ij})$ is normal if and only if $\sum_{i,j=1}^{n} |a_{ij}|^2 = \sum_{j=1}^{n} |\lambda_j|^2$, where λ_j are eigenvalues of A, $j = 1, 2, \ldots, n$.

For the adjacency matrix $A(S)$, we have

$$\sum_{i,j=1}^{n} |a_{ij}|^2 = \sum_{i,j=1}^{n} |\sigma(v_i, v_j)|^2 = \sum_{i,j=1}^{n} |\sigma(v_i, v_j)| = a.$$

Therefore (i) $\Longleftrightarrow$ (ii) follows.

(ii) $\Longrightarrow$ (iii). Assume

$$\sum_{j=1}^{n} |z_j|^2 = \sum_{j=1}^{n} (\Re z_j)^2 + \sum_{j=1}^{n} (\Im z_j)^2 = a.$$

By (i) of Lemma 2, we have

$$\sum_{j=1}^{n} (\Re z_j)^2 - \sum_{j=1}^{n} (\Im z_j)^2 = c_2^+ - c_2^-,$$

and therefore $\sum_{j=1}^{n} (\Re z_j)^2 = \frac{1}{2}(a + (c_2^+ - c_2^-))$.

(iii) $\Longrightarrow$ (ii). Assume

$$\sum_{j=1}^{n} (\Re z_j)^2 = \frac{1}{2}(a + (c_2^+ - c_2^-)).$$

Then by (i) of Lemma 2 $\sum_{j=1}^{n} (\Im z_j)^2 = \frac{1}{2}(a - (c_2^+ - c_2^-))$. Therefore,

$$\begin{aligned}\sum_{j=1}^{n} |z_j|^2 &= \sum_{j=1}^{n} (\Re z_j)^2 + \sum_{j=1}^{n} (\Im z_j)^2 \\ &= \frac{1}{2}(a + (c_2^+ - c_2^-)) + \frac{1}{2}(a - (c_2^+ - c_2^-)) = a.\end{aligned}$$

(iii) $\Longleftrightarrow$ (iv) is clear from (i) of Lemma 2. $\square$

A signed digraph S of order n is said to be unicyclic if it has n arcs and a unique cycle of length $r \leqslant n$. An application of Theorem 8 is the following result which gives a structural characterization of unicyclic signed digraphs to be normal.

Corollary 1. *Let S be a unicyclic signed digraph of order n. Then S is normal if and only if $S = C_n$ or $S = \mathbf{C_n}$.*

Proof. Let S be a unicyclic signed digraph of order and size n and with unique cycle of length $r \leqslant n$. The spec$(S) = \{0^{n-r}, e^{\frac{2\iota j\pi}{r}}\}$ or spec$(S) = \{0^{n-r}, e^{\frac{\iota(2j+1)\pi}{r}}\}$ according as S is cycle-balanced or non-cycle-balanced, where $\iota = \sqrt{-1}$ and $j = 0, 1, 2, \ldots, r-1$. The sum of the squares of absolute values of eigenvalues of S is r. By Theorem 8, clearly S is normal if and only if $r = n$, i.e., if and only if $S = C_n$, or $S = \mathbf{C_n}$. $\square$

The Kronecker product (strong product or conjunction) of two signed digraphs $S_1 = (V_1, \mathscr{A}_1, \sigma_1)$ and $S_2 = (V_2, \mathscr{A}_2, \sigma_2)$, denoted by $S_1 \otimes S_2$, is the signed digraph $(V_1 \times V_2, \mathscr{A}, \sigma)$, where arc set is the arc set of underlying unsigned digraphs and the sign function is defined by $\sigma((u_i, v_j), (u_k, v_l)) = \sigma_1(u_i, u_k)\sigma_2(v_j, v_l)$.

The following result connects the order and size of the Kronecker product of two signed digraphs in terms of those of constituent signed digraphs. The proof follows by definition.

Lemma 3. *Let $S_1(V_1, \mathscr{A}_1)$ and $S_2(V_2, \mathscr{A}_2)$ be two signed digraphs with $|V_i| = n_i$ and $|\mathscr{A}_i| = a_i$, $i = 1, 2$. Then $|V(S_1 \otimes S_2)| = n_1 n_2$ and $|\mathscr{A}(S_1 \otimes S_2)| = a_1 a_2$.*

The next result shows that the Kronecker product of two normal signed digraphs is normal.

Theorem 9. *If S_1 and S_2 are two normal signed digraphs, then $S_1 \otimes S_2$ is normal.*

Proof. Assume that $S_1(V_1, \mathscr{A}_1)$ and $S_2(V_2, \mathscr{A}_2)$ are two normal signed digraphs with $|V_i| = n_i$ and $|\mathscr{A}| = a_i$, $i = 1, 2$. Let z_{1i} and z_{2j}, with $1 \leqslant i \leqslant n_1$, $1 \leqslant j \leqslant n_2$ respectively be the eigenvalues of S_1 and S_2. By [[9], Theorem 4.5], the eigenvalues of $S_1 \otimes S_2$ are $z_{1i} z_{2j}$.

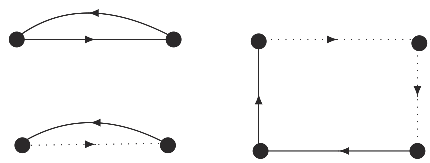
By Theorem 8, we have $\sum_{i=1}^{n_1} |z_{1i}|^2 = a_1$ and $\sum_{j=1}^{n_2} |z_{2j}|^2 = a_2$.

Therefore, $\sum_{i,j} |z_{1i} z_{2j}|^2 = \sum_i |z_{1i}|^2 \sum_j |z_{2j}|^2 = a_1 a_2$. By Theorem 8 and Lemma 3, the result follows. $\square$

Now we give the existence of cospectral normal signed digraphs.

Theorem 10. *For each positive integer $n \geqslant 4$, there exists a collection of n non-symmetric, cospectral normal signed digraphs.*

Proof. Consider the signed digraphs S_1 and S_2, shown in Fig. 5. It is clear that S_1 is non-cycle-balanced and S_2 is cycle-balanced. By Theorem 1, $\phi_{S_1}(x) = \phi_{S_2}(x) = x^4 - 1$. Therefore, $\mathrm{spec}(S_1) = \mathrm{spec}(S_2) = \{-1, 1, -\iota, \iota\}$. That is, S_1 and S_2 are cospectral. By Theorem 3.3, S_1 and S_2 are normal. Define $S^{[k]} = S_1 \otimes S_1 \otimes \cdots \otimes S_1 \otimes S_2 \otimes S_2 \otimes \cdots \otimes S_2$, where we take k copies of S_1 and $n-k$ copies of S_2. By Theorem 9, for each n, we have n cospectral normal signed digraphs $S^{[k]}, k = 1, 2, \ldots, n$ of order 4^n. S_1 and S_2 are non-symmetric implies $S^{[k]}$ is non-symmetric. $\square$



S_1 S_2

FIGURE 5. A pair of cospectral normal signed digraphs.

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References

[1] B. D. Acharya, *Spectral criterion for the cycle balance in networks*, J. Graph Theory, **4**, 1980, 1-11.

[2] B. D. Acharya, M. K. Gill and G. A. Patwardhan, *Quasicospectral graphs and digraphs*, National Symposium on Mathematical Modelling M. R. I. Allahabad: July 19-20, 1982.

[3] M. Acharya, *Quasi-cospectrality of graphs and digraphs: A creative review*, J. Comb. Inf. Syst. Sci. **37**, 2012, 241-256.

[4] M. A. Bhat and S. Pirzada, *On equienergetic signed graphs*, Discrete Appl. Math. **189**, 2015, 1-7.

[5] D. M. Cvetković, M. Doob and H. Sachs, *Spectra of Graphs*, Academic press, New York 1980.

[6] F. Esser and F. Harary, *Digraphs with real and Gaussian spectra*, Discrete Appl. Math. **2**, 1980, 113-124.

[7] R. A. Horn and C. R. Johnson. *Matrix analysis*, Cambridge University Press, Cambridge second edition, 2013.

[8] M. H. McAndrew, *On the product of directed graphs*, Proc. Amer. Math. Soc., **14**, 1963, 600-606.

[9] S. Pirzada and M. A. Bhat, *Energy of signed digraphs*, Discrete Appl. Math., **169**, 2014, 195-205.

[10] J. Rada, *Bounds for the energy of normal digraph*, Linear Multilinear Algebra, **60**, 2012, 323-332.

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