**Definition 5.4.** Let $G$ be a finitely generated group, acting on a set $A$. The growth degree of the $G$-action is the number

$$
\gamma = \sup_{w \in A} \limsup_{r \to \infty} \frac{\log |\{g(w) : l(g) \leq r\}|}{\log r}
$$

where $l(g)$ is the length of a group element with respect to some fixed finite generating set of $G$.

One can show, in the same way as before, that the growth degree $\gamma$ does not depend on the choice of the generating set of $G$.

**Proposition 5.10.** Suppose that a standard action of a group $G$ on $X^\omega$ is contracting. Then the growth degree of the action on $X^\omega$ is not greater than $\frac{\log |X|}{-\log \rho}$, where $\rho$ is the contraction coefficient of the action on $X^\omega$.

**Proof.** The statement is more or less classical. See, for instance the similar statements in [Gro81, BG00, Fra70].

Let $\rho_1$ be such that $\rho < \rho_1 < 1$. Then there exists $C > 0$ and $n \in \mathbb{N}$ such that for all $g \in G$ we have $l(g_{x_1x_2...x_n}) < \rho_1^n \cdot l(g) + C$.

Then cardinality of the set $B(w, r) = \{g(w) : l(g) \leq r\}$, where $w = x_1x_2... \in X^\omega$ is not greater than

$$|X|^n \cdot |\{B(x_{n+1}x_{n+2}...), \rho_1^n \cdot r + C\}|,$$

since the map $\sigma^n : x_1x_2... \mapsto x_{n+1}x_{n+2}...$ maps $B(w, r)$ into

$$B(x_{n+1}x_{n+2}... , \rho_1^n \cdot r + C)$$

and every point of $X^\omega$ has exactly $|X|^n$ preimages under $\sigma^n$. The map $\sigma^n$ is the $n$th iteration of the shift map $\sigma(x_1x_2...) = x_2x_3...$.

Let $k = \left[ \frac{\log r}{-n \log \rho_1} \right] + 1$. Then $\rho_1^{nk} \cdot r < 1$ and the number of the points in the ball $B(w, r)$ is not greater than

$$|X|^{nk} \cdot |B(\sigma^{nk}(w), R)|,$$

where

$$R = \rho_1^{nk} \cdot r + \rho_1^{n(k-1)} \cdot C + \rho_1^{n(k-2)} \cdot C + \cdots + \rho_1^{n} \cdot C + C < 1 + \frac{C}{1 - \rho_1^n}.$$

But $|B(u, R)|$ for all $u \in X^\omega$ is less than $K_1 = |S|^R$, where $S$ is the generating set of $G$ (we assume that $S = S^{-1} \ni 1$). Hence,

$$|B(w, r)| < K_1 \cdot |X|^{n\left( \frac{\log r}{-n \log \rho_1} + 1 \right)} = K_1 \cdot \exp \left( \frac{\log |X| \log r}{-n \log \rho_1} + n \log |X| \right) = K_2 \cdot r^{-\frac{\log |X|}{-\log \rho_1}},$$

where $K_2 = K_1 \cdot |X|^n$. Thus, the growth degree is not greater than $\frac{\log |X|}{-\log \rho_1}$ for every $\rho_1 \in (\rho, 1)$, so it is not greater than $\frac{\log |X|}{-\log \rho}$. \qed
Lemma 5.11. Let $\phi$ be a contracting virtual endomorphism of a $\phi$-simple infinite finitely generated group $G$. Then the contraction coefficient of its standard action is greater or equal to $1/\text{ind} \phi$.

Proof. Consider the standard action on the set $X^*$ for a standard basis $X$, containing the element $x_0 = \phi(1)1$. Then the parabolic subgroup $P(\phi) = \cap_{n\geq0} \text{Dom} \phi^n$ is the stabilizer of the word $w = x_0x_0x_0\ldots \in X^\omega$. The subgroup $P(\phi)$ has infinite index in $G$, otherwise $\cap_{g\in G} g^{-1} P g = \mathcal{C}(\phi)$ will have finite index, and $G$ will be not $\phi$-simple. Consequently, the $G$-orbit of $w$ is infinite. Then there exists an infinite sequence of generators $s_1, s_2, \ldots$ of the group $G$ such that the elements of the sequence

$$w, s_1(w), s_2s_1(w), s_3s_2s_1(w), \ldots$$

are pairwise different. This implies that the growth degree of the orbit $Gw$

$$\gamma = \limsup_{r\to\infty} \frac{|\{g(w) : l(g) \leq r\}|}{\log r}$$

is greater or equal to 1, thus the growth degree of the action of $G$ on $X^\omega$ is not less than 1, and by Proposition 5.10, $1 \leq \frac{\log |X|}{\log \rho}$.

Proposition 5.12. If there exists a faithful contracting action of a finitely-generated group $G$ then for any $\epsilon > 0$ there exists an algorithm of polynomial complexity of degree not greater than $\frac{\log |X|}{\log \rho} + \epsilon$ solving the word problem in $G$.

Proof. We assume that the generating set $S$ is symmetric (i.e., that $S = S^{-1}$) and contains all the restrictions of all its elements, so that always $l(g|_v)$ is not greater than $l(g)$.

We will denote by $F$ the free group generated by $S$ and for every $g \in F$ by $\hat{g}$ we denote the canonical image of $g$ in $G$.

Let $1 > \rho_1 > \rho$. Then $\rho_1 \cdot |X| > 1$, since by Lemma 5.11, $\rho \cdot |X| \geq 1$. There exist $n_0$ and $l_0$ such that for every word $v \in X^*$ of the length $n_0$ and every $g \in G$ of the length $\geq l_0$ we have

$$l(g|_v) < \rho_1^n l(g).$$

Assume that we know for every $g \in F$ of the length less than $l_0$ if $\hat{g}$ is trivial or not. Assume also that we know all the relations $g \cdot v = u \cdot h$ for all $g, l(g) \leq l_0$ and $v \in X^{n_0}$.

Then we can compute in $l(\hat{g})$ steps, for any $g \in F$ and $v \in X^n$, the element $h \in F$ and the word $u \in X^{n_0}$ such that $\hat{g} \cdot v = u \cdot \hat{h}$. If $v \neq u$ then we conclude that $\hat{g}$ is not trivial and stop the algorithm. If for all $v \in X^{n_0}$ we have $v = u$, then $\hat{g}$ is trivial if and only if all the obtained
restrictions \( \hat{h} = \hat{g}|_v \) are trivial. We know, whether \( \hat{h} \) is trivial if \( l(h) < l_0 \). We proceed further, applying the above computations for those \( h \), which have the length not less than \( l_0 \).

But \( l(h) < \rho_1^l(g) \), if \( l(g) \geq l_0 \). So on each step the length of the elements becomes smaller, and the algorithm stops in not more than \(-\log l(g)/\log \rho_1\) steps. On each step the algorithm branches into \(|X|\) algorithms. Thus, since \( \rho_1 \cdot |X| > 1 \), the total time is bounded by

\[
\begin{align*}
&\frac{l(g)}{\rho_1 \cdot |X| - 1} \left( (\rho_1 \cdot |X|)^{1 - \log l(g)/\log \rho_1} - 1 \right) = \\
&\frac{l(g)}{\rho_1 \cdot |X| - 1} \left( (\rho_1 \cdot |X|)^{\log |X|/\log \rho_1} - (\rho_1 \cdot |X|)^{-1} \right) = \\
&C_1 l(g) \left( \exp \left( \log l(g) \left( \frac{\log |X|}{\log \rho_1} - 1 \right) \right) - C_2 \right) = \\
=C_1 l(g)^{-\log |X|/\log \rho_1} - C_1 C_2 l(g),
\end{align*}
\]

where \( C_1 = \frac{\rho_1 \cdot |X|}{\rho_1 \cdot |X| - 1} \) and \( C_2 = (\rho_1 \cdot |X|)^{-1} \).

\[\square\]

References


Virtual endomorphisms of groups


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Metrizable ball structures

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Abstract. A ball structure is a triple \((X, P, B)\), where \(X, P\) are nonempty sets and, for any \(x \in X, \alpha \in P\), \(B(x, \alpha)\) is a subset of \(X\), \(x \in B(x, \alpha)\), which is called a ball of radius \(\alpha\) around \(x\). We characterize up to isomorphism the ball structures related to the metric spaces of different types and groups.

Following [1, 2], by ball structure we mean a triple \(B = (X, P, B)\), where \(X, P\) are nonempty sets and, for any \(x \in X, \alpha \in P\), \(B(x, \alpha)\) is a subset of \(X\), which is called a ball of radius \(\alpha\) around \(x\). It is supposed that \(x \in B(x, \alpha)\) for all \(x \in X, \alpha \in P\).

Let \(B_1 = (X_1, P_1, B_1)\) and \(B_2 = (X_2, P_2, B_2)\) be ball structures, \(f : X_1 \to X_2\). We say that \(f\) is a \(\succ\)-mapping if, for every \(\beta \in P_2\), there exists \(\alpha \in P_1\) such that

\[B_2(f(x), \beta) \subseteq f(B_1(x, \alpha))\]

for every \(x \in X_1\). If there exists a \(\succ\)-mapping of \(X_1\) onto \(X_2\), we write \(B_1 \succ B_2\).

A mapping \(f : X_1 \to X_2\) is called a \(\prec\)-mapping if, for every \(\alpha \in P_1\), there exists \(\beta \in P_2\) such that

\[f(B_1(x, \alpha)) \subseteq B_2(f(x), \beta)\]

for every \(x \in X_1\). If there exists an injective \(\prec\)-mapping of \(X_1\) into \(X_2\), we write \(B_1 \prec B_2\).

A bijection \(f : X_1 \to X_2\) is called an isomorphism between \(B_1\) and \(B_2\) if \(f\) is a \(\succ\)-mapping and \(f\) is a \(\prec\)-mapping.

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We say that a property $P$ of ball structures is a ball property if a ball structure $B$ has a property $P$ provided that $B$ is isomorphic to some ball structure with property $P$.

**Example 1.** Let $(X, d)$ be a metric space, $R^+ = \{ x \in R : x \geq 0 \}$. Given any $x \in X, r \in R^+$, put

$$B_d(x, r) = \{ y \in X : d(x, y) \leq r \}.$$ 

A ball structure $(X, R^+, B_d)$ is denoted by $B(X, d)$.

We say that a ball structure $B$ is metrizable if $B$ is isomorphic to $B(X, d)$ for some metric space $(X, d)$.

To obtain a characterization (Theorem 1) of metrizable ball structures, we need some definitions and technical results.

A ball structure $B = (X, P, B)$ is called connected if, for any $x, y \in X$, there exists $\alpha \in P$ such that $y \in B(x, \alpha)$, $x \in B(y, \alpha)$.

**Lemma 1.** Let $B_1 = (X_1, P_1, B_1)$ and $B_2 = (X_2, P_2, B_2)$ be ball structures and let $f$ be a $\prec$-mapping of $X_1$ onto $X_2$. If $B_1$ is connected, then $B_2$ is connected.

**Proof.** Given any $y, z \in X_1$, choose $\alpha \in P_1$ such that $y \in B_1(z, \alpha)$, $z \in B_1(y, \alpha)$. Since $f$ is a $\prec$-mapping, then there exists $\beta \in P_2$ such that $f(B_1(x, \alpha)) \subseteq B_2(f(x), \beta)$ for every $x \in X_1$. Hence, $f(y) \in B_2(f(z), \beta)$ and $f(z) \in B_2(f(y), \beta)$. Since $f(X_1) = X_2$, then $B_2$ is connected. \[\square\]

**Lemma 2.** Let $B_1 = (X_1, P_1, B_1)$ and $B_2 = (X_2, P_2, B_2)$ be ball structures and let $f$ be an injective $\succ$-mapping of $X_1$ into $X_2$. If $B_2$ is connected, then $B_1$ is connected.

**Proof.** Given any $y, z \in X_1$, choose $\beta \in P_2$ such that $f(y) \in B_2(f(z), \beta)$ and $f(z) \in B_2(f(y), \beta)$. Since $f$ is a $\succ$-mapping, then there exists $\alpha \in P_1$ such that $B_2(f(x), \beta) \subseteq f(B_1(x, \alpha))$ for every $x \in X_1$. Since $f$ is injective, then $z \in B_1(y, \alpha)$ and $y \in B_1(z, \alpha)$. Hence, $B_1$ is connected. \[\square\]

Let $B = (X, P, B)$ be a ball structure. For all $x \in X$, $\alpha \in P$, put

$$B^*(x, \alpha) = \{ y \in X : x \in B(y, \alpha) \}.$$

A ball structure $B^* = (X, P, B^*)$ is called dual to $B$. Note that $B^{**} = B$.

A ball structure $B$ is called symmetric if the identity mapping $i : X \rightarrow X$ is an isomorphism between $B$ and $B^*$. In other words, $B$ is symmetric if, for every $\alpha \in P$, there exists $\beta \in P$ such that $B(x, \alpha) \subseteq B^*(x, \beta)$ for every $x \in X$, and vice versa.
Lemma 3. Let $\mathcal{B}_1 = (X_1, P_1, B_1)$ and $\mathcal{B}_2 = (X_2, P_2, B_2)$ be ball structures, $f : X_1 \rightarrow X_2$. If $f$ is a $\prec$-mapping of $\mathcal{B}_1$ to $\mathcal{B}_2$, then $f$ is a $\prec$-mapping of $\mathcal{B}_1^*$ to $\mathcal{B}_2^*$. If $f$ is an isomorphism between $\mathcal{B}_1$ and $\mathcal{B}_2$, then $f$ is an isomorphism between $\mathcal{B}_1^*$ and $\mathcal{B}_2^*$.

Proof. Let $f$ be a $\prec$-mapping of $\mathcal{B}_1$ to $\mathcal{B}_2$ and let $\alpha \in P_1$. Choose $\beta \in P_2$ such that $f(B_1(x, \alpha)) \subseteq B_2(f(x), \beta)$ for every $x \in X_1$. Take any element $y \in B_1^*(x, \alpha)$. Then $x \in B_1(y, \alpha)$ and $f(x) \in B_2(f(y), \beta)$. Hence, $f(y) \in B_2^*(f(x), \beta)$ and $f(B_1^*(x, \alpha)) \subseteq B_2^*(f(x), \beta)$. It means that $f$ is a $\prec$-mapping of $\mathcal{B}_1^*$ to $\mathcal{B}_2^*$.

Suppose that $f$ is an isomorphism between $\mathcal{B}_1$ and $\mathcal{B}_2$. By the first statement, $f$ is a $\prec$-mapping of $\mathcal{B}_1^*$ to $\mathcal{B}_2^*$ and $f^{-1}$ is a $\prec$-mapping of $\mathcal{B}_2^*$ to $\mathcal{B}_1^*$. It follows that $f$ is an isomorphism between $\mathcal{B}_1^*$ and $\mathcal{B}_2^*$. $\square$

Lemma 4. Let $\mathcal{B}_1 = (X_1, P_1, B_1)$ and $\mathcal{B}_2 = (X_2, P_2, B_2)$ be isomorphic ball structures. If $\mathcal{B}_1$ is symmetric, then $\mathcal{B}_2$ is symmetric.

Proof. Let $f : X_1 \rightarrow X_2$ be an isomorphism between $\mathcal{B}_1$ and $\mathcal{B}_2$. Denote by $i_1 : X_1 \rightarrow X_1$ and $i_2 : X_2 \rightarrow X_2$ the identity mappings. Clearly, $f^{-1}$ is an isomorphism between $\mathcal{B}_2$ and $\mathcal{B}_1$. By Lemma 3, $f$ is an isomorphism between $\mathcal{B}_1^*$ and $\mathcal{B}_2^*$. By assumption, $i_1$ is an isomorphism between $\mathcal{B}_1$ and $\mathcal{B}_1^*$. Since $i_2 = f i_1 f^{-1}$, then $i_2$ is an isomorphism between $\mathcal{B}_2$ and $\mathcal{B}_2^*$. $\square$

A ball structure $\mathcal{B} = (X, P, B)$ is called multiplicative if, for any $\alpha, \beta \in P$, there exists $\gamma(\alpha, \beta) \in P$ such that

$$B(B(x, \alpha), \beta) \subseteq B(x, \gamma(\alpha, \beta))$$

for every $x \in X$. Here, $B(A, \alpha) = \bigcup_{a \in A} B(a, \alpha)$ for any $A \subseteq X, \alpha \in P$.

Lemma 5. If a ball structure $\mathcal{B} = (X, P, B)$ is multiplicative, then $\mathcal{B}^*$ is multiplicative.

Proof. Given any $\alpha, \beta \in P$, choose $\gamma(\alpha, \beta)$ such that $B(B(x, \alpha), \beta) \subseteq B(x, \gamma(\alpha, \beta))$. Take any element $z \in B^*(B^*(x, \alpha), \beta)$ and pick $y \in B^*(x, \alpha)$ such that $z \in B^*(y, \beta)$. Then $x \in B(y, \alpha)$ and $y \in B(z, \beta)$, so $x \in B(B(z, \beta), \alpha)$. Since $B(B(z, \beta), \alpha) \subseteq B(z, \gamma(\beta, \alpha))$, then $x \in B(z, \gamma(\beta, \alpha))$. Hence, $B^*(B^*(x, \alpha), \beta) \subseteq B^*(x, \gamma(\beta, \alpha))$ and $\mathcal{B}^*$ is multiplicative. $\square$

Lemma 6. Let $\mathcal{B}_1 = (X_1, P_1, B_1)$ and $\mathcal{B}_2 = (X_2, P_2, B_2)$ be isomorphic ball structures. If $\mathcal{B}_1$ is multiplicative, then $\mathcal{B}_2$ is multiplicative.
Proof. Denote by $f_1 : X_1 \to X_2$ the isomorphism between $B_1$ and $B_2$. Fix any $\beta_1, \beta_2 \in P_2$. Since $f$ is a bijection, it suffices to prove that there exists $\beta \in P_2$ such that

$$B_2(B_2(f(x), \beta_1), \beta_2) \subseteq B_2(f(x), \beta)$$

for every $x \in X_1$.

Since $f$ is a $\succ$-mapping, then there exist $\alpha_1, \alpha_2 \in P_1$ such that

$$B_2(f(x), \beta_1) \subseteq f(B_1(x, \alpha_1)), B_2(f(x), \beta_2) \subseteq f(B_1(x, \alpha_2))$$

for every $x \in X_1$.

Since $B_1$ is multiplicative, then there exists $\alpha \in P_1$ such that

$$B_1(B_1(x, \alpha_1), \alpha_2) \subseteq B_1(x, \alpha)$$

for every $x \in X_1$.

Since $f$ is a $\prec$-mapping, then there exists $\beta \in P_2$ such that

$$f(B_1(x, \alpha)) \subseteq B_2(f(x), \beta)$$

for every $x \in X_1$.

Now fix $x \in X_1$ and take any element $f(z) \in B_2(B_2(f(x), \beta_1), \beta_2)$. Pick $f(y) \in B_2(f(x), \beta_1)$ with $f(z) \in B_2(f(y), \beta_2)$. Then $y \in B_1(x, \alpha_1)$, $z \in B_1(y, \alpha_2)$ and $z \in B_1(B_1(x, \alpha_1), \alpha_2)$. Hence, $z \in B_1(x, \alpha)$ and $f(z) \in B_2(f(x), \beta)$.

For an arbitrary ball structure $B = (X, P, B)$, we define a preordering $\leq$ on the set $P$ by the rule

$$\alpha \leq \beta \text{ if and only if } B(x, \alpha) \subseteq B(x, \beta)$$

for every $x \in X$. A subset $P'$ of $P$ is called cofinal if, for every $\alpha \in P$, there exists $\beta \in P'$ such that $\alpha \leq \beta$. A cofinality $\text{cf} B$ of $B$ is a minimum of cardinalities of cofinal subsets of $P$. Thus, $\text{cf} B \leq \aleph_0$ if and only if there exists a cofinal sequence $< \alpha_n >_{n \in \omega}$ in $P$ such that $\alpha_0 \leq \alpha_1 \leq \ldots \leq \alpha_n \leq \ldots$

Lemma 7. If the ball structures $B_1 = (X_1, P_1, B_1)$ and $B_2 = (X_2, P_2, B_2)$ are isomorphic, then $\text{cf} B_1 = \text{cf} B_2$.

Proof. Let $f : X_1 \to X_2$ be an isomorphism between $B_1$ and $B_2$ and let $P_1'$ be a cofinal subset of $P_1$. Since $f$ is a $\succ$-mapping, then there exists a mapping $h_1 : P_1 \to P_1'$ such that $B_2(f(x), \beta) \subseteq f(B_1(x, h_1(\beta)))$ for any $x \in X_1$, $\beta \in P_2$. Since $f$ is a $\prec$-mapping, then there exists a mapping $h_2 : P_1' \to P_2$ such that $f(B_1(x, \alpha)) \subseteq B_2(f(x), h_2(\alpha))$ for any $x \in X_1$, $\alpha \in P_1'$. From the construction of $h_1, h_2$ we conclude that $h_2(P_1')$ is a cofinal subset of $P_2$. Hence, $\text{cf} B_2 \leq \text{cf} B_1$. 

\Box
Theorem 1. A ball structure $B = (X, P, B)$ is metrizable if and only if $B$ is connected symmetric multiplicative and $\text{cf}B \leq \aleph_0$.

Proof. First suppose that $B$ is isomorphic to $B(X, d)$ for an appropriate metric space $(X, d)$. Obviously, $B(X, d)$ is connected symmetric multiplicative and $\text{cf}B \leq \aleph_0$. By Lemma 1, 4, 6, 7 $B$ has the same properties.

Now assume that $B$ is connected symmetric multiplicative and $\text{cf}B \leq \aleph_0$. Let $x, y, z \in X$. It follows from the proof of Theorem 1 that, for every ball structure $B = (X, P, B)$, there exists an integer $n \geq \omega$ for an appropriate $n \in \omega$ is a nondecreasing cofinal sequence in $P$ and $B(B(x, \beta_n), \beta_m) \subseteq B(x, \beta_{n+m})$ for all $x \in X, n, m \in N$. $B(x, \beta_n), \beta_m) \subseteq B(x, \beta_{n+m})$. Since $y \in B(z, \beta_n)$ and $x \in B(y, \beta_n)$, then $x \in B(B(z, \beta_m), \beta_n) \subseteq B(z, \beta_{n+m})$.

Hence, $d(x, z) \leq n + m$.

Consider the ball structure $B(X, d)$ and note that $B_d(x, n) = B(x, \beta_n) \cap B^*(x, \beta_n)$. Since $B$ is symmetric, then the identity mapping of $X$ is an isomorphism between $B$ and $B(X, d)$.

Remark 1. A metric $d$ on a set $X$ is called integer if $d(x, y)$ is an integer number for all $x, y \in X$. It follows from the proof of Theorem 1 that, for every metrizable ball structure $B = (X, P, B)$, there exists an integer metric $d$ on $X$ such that $B$ and $B(X, d)$ are isomorphic.

Remark 2. Let $B = (X, P, B)$ be an arbitrary ball structure. Consider a metric $d$ on $X$ defined by the rule $d(x, x) = 0$ and $d(x, y) = 1$ for all distinct elements of $X$. Then the identity mapping $i : X \to X$ is a $\prec$-mapping of $B$ onto $B(X, d)$. In particular, for every ball structure $B$, there exists a metric space $(X, d)$ such that $B \prec B(X, d)$. 

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Remark 3. Let $B = (X, P, B)$ be a connected multiplicative ball structure, cf $\mathcal{B} \leq \aleph_0$. Repeating arguments of Theorem 1, we can prove that there exists a metric $d$ on $X$ such that the identity mapping $i : X \to X$ is a $\prec$-mapping of $B(X, d)$ onto $B$.

Question 1. Characterize the ball structure $B = (X, P, B)$, which admit a metric $d$ on $X$ such that the identity mapping $i : X \to X$ is a $\prec$-mapping of $B(X, d)$ onto $B$.

By Remark 2, every ball structure can be strengthened to some metrizable ball structure, so Question 1 asks about ball structure, which can be weakened to metrizable.

Example 2. Let $Gr = (V, E)$ be a connected graph with a set of vertices $V$ and a set of edges $E$, $E \subseteq V \times V$. Endow $V$ with a path metric $d$, where $d(x, y)$, $x, y \in V$ is a length of the shortest path between $x$ and $y$. Denote by $B(Gr)$ the ball structure $B(V, d)$. Obviously, $B(Gr)$ is metrizable.

Our next target is a description of the ball structures, isomorphic to $B(Gr)$ for an appropriate graph $Gr$.

Let $B = (X, P, B)$ be an arbitrary ball structure, $\alpha \in P$. We say that a finite sequence $x_0, x_1, \ldots, x_n$ of elements of $X$ is an $\alpha$-path of length $n$ if $x_{i-1} \in B(x_i, \alpha)$, $x_i \in B(x_{i-1}, \alpha)$ for every $i \in \{1, 2, \ldots, n\}$. A ball structure $B$ is called an $\alpha$-path connected if, for every $\beta \in P$, there exists $\mu(\beta) \in \omega$ such that $x \in B(y, \beta)$, $y \in B(x, \beta)$ imply that there exists an $\alpha$-path of length $\leq \mu(\beta)$ between $x$ and $y$. Note that $B(Gr)$ is 1-path connected for every connected graph $Gr$.

A ball structure $B = (X, P, B)$ is called path connected if $B$ is $\alpha$-path connected for some $\alpha \in P$.

Lemma 8. Let $B_1 = (X_1, P_1, B_1)$ and $B_2 = (X_2, P_2, B_2)$ be isomorphic ball structures. If $B_1$ is path connected, then $B_2$ path connected.

Proof. Let $f : X_1 \to X_2$ be an isomorphism between $B_1$ and $B_2$. Choose $\alpha \in P_1$ such that $B_1$ is $\alpha$-path connected and fix a corresponding mapping $\mu : P_1 \to \omega$. Since $f$ is a $\prec$-mapping, then there exists $\beta \in P_2$ such that

$$f(B_1(x, \alpha)) \subseteq B_2(f(x), \beta)$$

for every $x \in X_1$. Since $f$ is a $\prec$-mapping, then there exists a mapping $h : P_2 \to P_1$ such that

$$B_2(f(x), \lambda) \subseteq f(B_1(x, h(\lambda)))$$

for any $x \in X_1, \lambda \in P_2$. 

Fix any \( \lambda \in P_2 \) and suppose that

\[
f(x) \in B_2(f(y), \lambda), \quad f(y) \in B_2(f(x), \lambda).
\]

Since \( f \) is a bijection, then \( x \in B_1(y, h(\lambda)), \quad y \in B_1(x, h(\lambda)) \). Since \( B_1 \) is \( \alpha \)-path connected, then there exists an \( \alpha \)-path \( x = x_0, x_1, \ldots, x_m = y \) of length \( \leq \mu(h(\lambda)) \). Then \( f(x) = f(x_0), f(x_1), \ldots, f(x_m) = f(y) \) is a \( \beta \)-path of length \( \leq \mu(h(\lambda)) \) between \( f(x) \) and \( f(y) \).

**Theorem 2.** For every ball structure \( B \), the following statements are equivalent

(i) \( B \) is metrizable and path connected;

(ii) \( B \) is isomorphic to a ball structure \( B(Gr) \) for some connected graph \( Gr \).

**Proof.** (ii)\( \Rightarrow \) (i). Clearly, \( B(Gr) \) is metrizable and path connected. Hence, \( B \) is metrizable and path connected by Lemma 8.

(i)\( \Rightarrow \) (ii). Fix a path connected metric space \((X, d)\) such that \( B \) is isomorphic to \( B(X, d) \). Then there exists \( m \in \omega \) such that \((X, d)\) is \( m \)-path connected. Consider a graph \( Gr = (X, E) \) with the set \( E \) of edges defined by the rule

\[
(x, y) \in E \text{ if and only if } x \neq y \text{ and } d(x, y) \leq m.
\]

Since \( B(X, d) \) is path connected, then the graph \( Gr \) is connected.

Let \( d' \) be a path metric on the graph \( Gr \). By assumption, for every \( n \in \omega \), there exists \( \mu(n) \in \omega \) such that \( d(x, y) \leq n \) implies that there exists a \( m \)-path of length \( \leq \mu(n) \) in \((X, d)\) between \( x \) and \( y \). Hence, \( d(x, y) \leq n \) implies \( d'(x, y) \leq \mu(n) \). On the other side, \( d'(x, y) \leq k \) implies that \( d(x, y) \leq km \). Therefore, the identity mapping of \( X \) is an isomorphism between the ball structures \( B(X, d) \) and \( B(Gr) \). \( \square \)

**Example 3.** Let \( X = \{2^n : n \in \omega \}, \) \( d(x, y) = |x - y| \) for any \( x, y \in X \). By Theorem 2, there are no connected graphs \( Gr \) such that \( B(X, d) \) is isomorphic to \( B(Gr) \).

**Example 4.** Let \( d \) be an euclidean metric on \( \mathbb{R}^n \). By Theorem 2, there exists a connected graph \( Gr_n = (\mathbb{R}^n, E_n) \) such that \( B(\mathbb{R}^n, d) \) is isomorphic to \( B(Gr_n) \).

By Remark 2, for every ball structure \( B = (X, P, B) \), there exists a connected graph \( Gr = (X, E), E = \{(x, y) : x, y \in X, x \neq y \} \) such that the identity mapping \( i : X \to X \) is a \( \succ \)-mapping of \( B(Gr) \) onto \( B \).

**Question 2.** Characterize the ball structure, which admit a \( \succ \)-bijection to the ball structure \( B(Gr) \) for an appropriate graph \( Gr \).
A metric \(d\) on a set \(X\) is called non-Archimedean if
\[
d(x, z) \leq \max \{d(x, y), d(y, z)\}
\]
for all \(x, y, z \in X\). The following definitions will be used to describe the ball structures isomorphic to \(B(X, d)\) for an appropriate non-Archimedean metric space \((X, d)\).

Let \(B = (X, P, B)\) be an arbitrary ball structure, \(x \in X\), \(\alpha \in P\). We say that a ball \(B(x, \alpha)\) is a cell if \(B(y, \alpha) = B(x, \alpha)\) for every \(y \in B(x, \alpha)\). If \((X, d)\) is a non-Archimedean metric space, then each ball \(B(x, r)\), \(x \in X\), \(r \in \mathbb{R}^+\) is a cell.

Given any \(x \in X\), \(\alpha \in P\), denote
\[
\beta^c(x, \alpha) = \{y \in X : \text{there exists an } \alpha - \text{path between } x \text{ and } y\}.
\]
A ball structure \(B^c = (X, P, B^c)\) is called a cellularization of \(B\). Note that each ball \(B^c(x, \alpha)\) is a cell.

We say that a ball structure \(B\) is cellular if the identity mapping \(i : X \to X\) is an isomorphism between \(B\) and \(B^c\). In other words, \(B\) is cellular if and only if, for every \(\alpha \in P\), there exists \(\beta \in P\) such that \(B(x, \alpha) \subseteq B^c(x, \beta)\) for every \(x \in X\) and, for every \(\beta \in P\), there exists \(\alpha \in P\) such that \(B^c(x, \beta) \subseteq B(x, \alpha)\) for every \(x \in X\).

A ball structure \(B = (X, P, B)\) is called directed if, for any \(\alpha, \beta \in P\), there exists \(\gamma \in P\) such that \(\alpha \leq \gamma, \beta \leq \gamma\).

**Lemma 9.** If \(B = (X, P, B)\) is a directed symmetric ball structure, then the identity mapping \(i : X \to X\) is a \(\prec\)-mapping of \(B\) onto \(B^c\).

**Proof.** Given any \(\alpha \in P\), choose \(\beta, \gamma \in P\) such that
\[
B(x, \alpha) \subseteq B^c(x, \beta) \subseteq B(x, \gamma)
\]
for every \(x \in X\). Since \(B\) is directed, we may assume that \(\beta \leq \gamma\). Take any element \(y \in B(x, \alpha)\). Then \(x \in B(y, \beta) \subseteq B(y, \gamma)\). Thus, \(y \in B(x, \gamma), x \in B(y, \gamma)\). Hence, there exists a \(\beta\)-path of length \(\leq 1\) between \(x\) and \(y\). It means that \(y \in B^c(x, \gamma)\), so \(B(x, \alpha) \subseteq B^c(x, \gamma)\). \(\square\)

**Lemma 10.** Let \(B_1 = (X_1, P_1, B_1)\) and \(B_2 = (X_2, P_2, B_2)\) be ball structures. If \(f : X_1 \to X_2\) is a \(\prec\)-mapping of \(B_1\) to \(B_2\), then \(f\) is a \(\prec\)-mapping of \(B_1^c\) to \(B_2^c\). If \(f\) is an isomorphism between \(B_1\) and \(B_2\), then \(f\) is an isomorphism between \(B_1^c\) and \(B_2^c\).

**Proof.** Given any \(\alpha \in P_1\), choose \(\beta \in P_2\) such that \(f(B_1(x, \alpha)) \subseteq B_2(f(x), \beta)\) for every \(x \in X\). Take any \(y \in B_1^c(x, \alpha)\) and choose an \(\alpha\)-path \(x = x_0, x_1, \ldots, x_n = y\) between \(x\) and \(y\). Then
\[
f(x) = f(x_0), f(x_1), \ldots, f(x_n) = f(y)
\]
is an $\beta$-path between $f(x)$ and $f(y)$. Hence, $f(y) \in B_2^c(f(x), \beta)$ and $f(B_1^c(x, \alpha)) \subseteq B_2^c(f(x), \beta)$ for every $x \in X_1$.

Suppose that $f$ is an isomorphism between $B_1$ and $B_2$. By the first statement, $f$ is a $\prec$-mapping of $B_1^c$ to $B_2^c$ and $f^{-1}$ is a $\prec$-mapping of $B_2^c$ to $B_1^c$. Hence, $f$ is an isomorphism between $B_1^c$ and $B_2^c$.

Lemma 11. Let $B_1 = (X_1, P_1, B_1)$ and $B_2 = (X_2, P_2, B_2)$ be isomorphic ball structures. If $B_1$ is cellular, then $B_2$ is cellular.

Proof. Let $f : X_1 \to X_2$ be an isomorphism between $B_1$ and $B_2$. Denote by $i_1 : X_1 \to X_1$ and $i_2 : X_2 \to X_2$ the identity mappings. Clearly, $f^{-1}$ is an isomorphism between $B_2$ and $B_1$. By the Lemma 10, $f$ is an isomorphism between $B_1^c$ and $B_2^c$. By assumption, $i_1$ is an isomorphism between $B_1$ and $B_1^c$. Since $i_2 = f i_1 f^{-1}$, then $i_2$ is an isomorphism between $B_2$ and $B_2^c$.

Theorem 3. For every ball structure $B$, the following statements are equivalent

(i) $B$ is metrizable and cellular;
(ii) there exists a non-Archimedean metric space $(X, d)$ such that $B$ is isomorphic to $B(X, d)$.

Proof. (ii) $\Rightarrow$ (i). Clearly, $B(X, d)$ is metrizable and cellular. Hence, $B$ is metrizable and cellular by Lemma 11.

(i) $\Rightarrow$ (ii). Fix a metric space $(X, d')$ such that $B(X, d')$ is cellular and isomorphic to $B$. Define a mapping $d : X \times X \to \omega$ by the rule

$$d(x, y) = \min\{m \in \omega : y \in B^c(x, m)\}.$$

Obviously, $d(x, x) = 0$ and $d(x, y) = d(y, x)$ for all $x, y \in X$.

Let $x, y, z \in X$ and let $d(x, y) = m, d(y, z) = n, m \leq n$. Then $y \in B^c(x, m), z \in B^c(y, n)$. It follows that there exists a $n$-path between $x$ and $z$. Hence, $z \in B^c(x, n)$ and $d(x, z) \leq n$. Thus, we have proved that $d$ is a non-Archimedean metric on $X$.

Since $d(x, y) \leq d'(x, y)$, then the identity mapping $i : X \to X$ is a $\prec$-mapping of $B(X, d)$ to $B(X, d')$. Since $B(X, d')$ is cellular, then there exists a mapping $h : \omega \to \omega$ such that $B^c(x, m) \subseteq B(x, h(m))$ for all $x \in X, m \in \omega$. Hence, $i$ is a $\succ$-mapping of $B(X, d)$ to $B(X, d')$. Hence, $B(X, d)$ and $B(X, d')$ are isomorphic.

By Remark 2, for every ball structure $B = (X, P, B)$, there exists a non-Archimedean metric $d$ on $X$ such that the identity mapping of $X$ is a $\succ$-mapping of $B(X, d)$ to $B$. 

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Lemma 12. For every metric space \((X,d)\), there exists a family \(\{P_n : n \in \omega\}\) of partitions of \(X\) with the following properties

(i) every partition \(P_{n+1}\) is an enlargement of \(P_n\), i.e. every cell of the partition \(P_{n+1}\) is a union of some cells of the partition \(P_n\);

(ii) there exists a function \(f : \omega \to \omega\) such that, for every \(C \in P_n\) and every \(x \in C, C \subseteq B(x, f(n))\);

(iii) for any \(x, y \in X\), there exists \(n \in \omega\) such that \(x, y\) are in the same cell of the partition \(P_n\).

Proof. Fix any well-ordering \(\{x_\alpha : \alpha < \gamma\}\) of \(X\). Choose a subset \(Y_0 \subseteq X\), \(x_0 \in Y_0\) such that the family \(\{B(y, 1) : y \in Y_0\}\) is disjoint and maximal. For every \(x \in X\), pick a minimal element \(f_0(x) \in Y_0\) such that \(B(x, 1) \cap B(f_0(x), 1) \neq \emptyset\). Put \(H(x, 1) = \{z \in X : f_0(z) = f_0(x)\}\) and note that the family \(\{H(y, 1) : y \in Y_0\}\) is a partition of \(X\). If \(x, z \in H(y, 1)\), then \(d(x, y) \leq 2, d(x, z) \leq 2\). Therefore, \(H(y, 1) \subseteq B(x, 4)\) for every \(x \in H(y, 1)\). Put \(P_0 = \{H(y, 1) : y \in Y_0\}, f(0) = 4\).

Assume that the partitions \(P_0, P_1, \ldots, P_{n-1}\) have been constructed and the values \(f(0), f(1), \ldots, f(n-1)\) have been determined. Choose a subset \(Y_n \subseteq X, x_0 \in Y_n\) such that the family \(\{B(y, n+1) : y \in Y_n\}\) is disjoint and maximal. Define a mapping \(f_n : X \to Y_n\) inductively such that \(f_n\) is constant on each cell of the partition \(P_{n-1}\). Put \(f_n(x) = x_0\) for every \(x \in X\) such that \(H(x, n) \cap B(x_0, n+1) \neq \emptyset\). Then take the minimal element \(x \in X\) such that \(f_n(x)\) is not determined. Choose the minimal element \(y \in Y_n\) such that \(B(x, n+1) \cap B(y, n+1) \neq \emptyset\). Put \(f_n(x) = y\) and \(f_n(z) = y\) for every \(z \in H(x, n)\). After this transfinite procedure, we denote \(H(x, n+1) = \{z \in X : f_n(z) = f_n(x)\}\). Put \(P_n = \{H(y, n+1) : y \in Y_n\}\). Then \(P_n\) is a partition of \(X\) and each cell of \(P_n\) is a union of some cells of \(P_{n-1}\). Thus, (i) is satisfied.

If \(z \in H(y, n+1)\), then \(d(z, y) \leq f(n-1) + 2(n+1)\). Hence, to satisfy (ii), put \(f(n) = 2(f(n-1) + 2(n+1))\).

At last, given any \(x, y \in X\), choose \(m \in \omega\) such that \(d(x_0, x) \leq m+1, d(x_0, y) \leq m+1\). Thus \(x, y\) are in the same cell of the partition \(P_m\) and we have verified (iii). \(\square\)

Theorem 4. For every metric space \((X,d)\), there exists a non-Archimedian metric \(d'\) on \(X\) such that the identity mapping \(i : X \to X\) is a \(\prec\)-mapping of \(B(X,d')\) to \(B(X,d)\).

Proof. Fix a family \(\{P_n : n \in \omega\}\) of partitions of \(X\), satisfying (i), (ii), (iii) from Lemma 12. Define a mapping \(d' : X \times X \to \omega\) by the rule

\[
d'(x, y) = \min\{n : x \text{ and } y \text{ are in the same cell of } P_n\}.
\]
By (iii), $d'$ is well defined. By (i), $d'$ is a non-Archimedian metric. By (ii), the identity mapping of $X$ is a $\prec$-mapping of $B(X, d')$ onto $B(X, d)$.

Now we consider non-metrizable versions of Lemma 12 and Theorem 4.

**Lemma 13.** Let $B = (X, P, B)$ be a directed symmetric multiplicative ball structure. Then there exists a family $\{P_\alpha : \alpha \in P\}$ of partitions of $X$ such that

(i) for every $\alpha \in P$, there exists $\beta \in P$ such that $C \subseteq B(x, \beta)$ for every $C \in P_\alpha$ and every $x \in C$.

Moreover, if $B$ is connected then

(ii) for any $x, y \in X$, there exists $\alpha \in P$ such that $x, y$ are in the same cell of the partition $P_\alpha$.

**Proof.** Fix any well-ordering of $X$ and denote by $x_0$ its minimal element. Fix $\alpha \in P$ and choose a subset $Y \subseteq X$, $x_0 \in Y$ such that the family $\{B(y, \alpha) : y \in Y\}$ is disjoint and maximal. For every $x \in X$, pick a minimal element $f(x) \in Y$ such that $B(x, \alpha) \cap B(f(x), \alpha) \neq \emptyset$. Put $H(x, \alpha) = \{z \in X : f(z) = f(x)\}$. Then the family $P_\alpha = \{H(y, \alpha) : y \in Y\}$ is a partition of $X$.

Since $B$ is directed and symmetric, then there exists $\alpha' > \alpha$ such that $y \in B(x, \alpha)$ implies $x \in B(y, \alpha')$.

Fix $x \in X$ and take $x' \in B(x, \alpha) \cap B(f(x), \alpha)$. Then $x, x', f(x)$ is an $\alpha'$-path. Hence, for every $z \in H(x, \alpha)$, we can find an $\alpha'$-path of length 4 between $x$ and $z$. Using multiplicativity of $B$, choose $\beta \in P$ such that $y_4 \in B(y_0, \beta)$ for every $\alpha'$-path $y_0, y_1, y_2, y_3, y_4$ in $X$. Then $H(x, \alpha) \subseteq B(x, \beta)$.

Suppose that $B$ is connected and $x, y \in X$. Since $B$ is directed, then there exists $\alpha \in P$ such that $x_0 \in B(x, \alpha)$, $x_0 \in B(y, \alpha)$. Hence, $x, y$ belong to the cell $H(x_0, \alpha)$ of the partition $P_\alpha$.

**Theorem 5.** If a ball structure $B = (X, P, B)$ is directed symmetric and multiplicative, then there exists a cellular ball structure $B' = (X, P, B')$ such that the identity mapping of $X$ is a $\prec$-mapping of $B'$ onto $B$. Moreover, if $B$ is connected, then $B'$ is connected.

**Proof.** Use the family of the partitions $\{P_\alpha : \alpha \in P\}$ from Lemma 13 and put $B'(x, \alpha) = H(x, \alpha)$. Clearly, each ball $B'(x, \alpha)$ is a cell. By (i), the identity mapping of $X$ is a $\prec$-mapping of $B'$ onto $B$. If $B$ is connected, then $B'$ is connected by (ii).
Example 5. Let $G$ be a group and let $\text{Fin}_c(G)$ be a family of all finite subsets of $G$ containing the identity $e$. Given any $g \in G$, $F \in \text{Fin}_c(G)$, put $B(g, F) = Fg$. A ball structure $\mathbf{B}(G) = (G, \text{Fin}_c(G), B)$ is denoted by $\mathbf{B}(G)$. It is easy to show, that $\mathbf{B}(G)$ is directed connected symmetric and multiplicative.

Now we apply the above results to the ball structures of groups.

Theorem 6. Let $G$ be a group. Then a ball structure $\mathbf{B}(G)$ is metrizable if and only if $|G| \leq \aleph_0$.

Proof. Apply Theorem 1.

Theorem 7. For every group $G$, the following statements are equivalent

(i) $G$ is finitely generated;

(ii) $\mathbf{B}(G)$ is isomorphic to $\mathbf{B}(Gr)$ for some connected graph $Gr$.

Proof. (i)$\Rightarrow$(ii). Let $S$ be a finite set of generators of $G$. Consider a Cayley graph $Gr = (G, E)$ of $G$ determined by $S$. By definition, $(x, y) \in E$ if and only if $x \neq y$ and $x = ty$ for some $t \in S \cup S^{-1}$. Clearly, the identity mapping of $G$ is an isomorphism between $\mathbf{B}(G)$ and $\mathbf{B}(Gr)$.

(ii)$\Rightarrow$(i). By Theorem 2, there exists $F \in \text{Fin}$ such that $\mathbf{B}(G)$ is $F$-path connected. In particular, for every $g \in G$, there exists a $F$-path between $e$ and $g$. Hence, $F$ generates $G$.

A group $G$ is called locally finite if every finite subset of $G$ generates a finite subgroup.

Theorem 8. Let $G$ be a group. Then a ball structure $\mathbf{B}(G)$ is cellular if and only if $G$ is locally finite.

Proof. Let $G$ be locally finite. Denote by $\text{Fin}_s$ the family of all finite subgroups of $G$. Then $\text{Fin}_s$ is cofinal in $\text{Fin}$ and each ball $B(g, F)$, $F \in \text{Fin}_s$ is a cell. Hence, $\mathbf{B}(G)$ is cellular.

Assume that $\mathbf{B}(G)$ is cellular. Note that $B^c(e, F) = gpF$ for every $F \in \text{Fin}$, where $gpF$ is a subgroup of $G$ generated by $F$. Since $\mathbf{B}$ is isomorphic to $\mathbf{B}^c$, then each ball $B^c(g, F)$ is finite. In particular, $gpF$ is finite for every $F \in \text{Fin}$.

Remark 4. Let $G_1$, $G_2$ be countable locally finite group. By [2, Theorem 4], $\mathbf{B}(G_1) \succ \mathbf{B}(G_2)$ and $\mathbf{B}(G_1) \prec \mathbf{B}(G_2)$. By [2, Theorem 5], $\mathbf{B}(G_1)$ and $\mathbf{B}(G_2)$ are isomorphic if and only if, for every finite subgroup $F$ of $G_1$, there exists a finite subgroup $H$ of $G_2$ such that $|F|$ is a divisor of $|H|$, and vice versa. A problem of classification up to an isomorphism of ball structures of uncountable locally finite groups is open.
Theorem 9. For every countable group $G$, there exists a non-Archimedean metric $d$ on $G$ with the following property

(i) for each $n \in \omega$, there exists $F \in Fin$ such that $d(x, y) \leq n$ implies $x \in F y$.

Proof. Apply Theorem 6 and Theorem 4.

Theorem 10. For every group $G$, there exists a cellular ball structure $B' = (G, Fin, B')$ such that the identity mapping of $G$ is a $\prec$-mapping of $B'$ onto $B(G)$.

Proof. Apply Theorem 5.

Question 3. Characterize the ball structures isomorphic to the ball structures of groups.

M.Zarichnyi has pointed out that Theorem 1 has a counterpart in the asymptotic topology [3]. This theorem answers the Open Question 1 from [4]. The results of this paper was announced in [5].

References


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